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HU–WASHIZU VARIATIONAL PRINCIPLE IN PROBLEMS OF STABILITY OF NON-THIN ANISOTROPIC CYLINDRICAL SHELLS MADE OF MODERN COMPOSITE MATERIALS IN SPATIAL FORMULATION

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ABSTRACT

In this study, the Hu–Washizu variational principle was employed to derive a three-dimensional system of partial differential equations, which governs the stability of an anisotropic body. This system was expressed in the cylindrical coordinate system. The analytical Bubnov–Galerkin method was employed to reduce it to a one-dimensional one. The solution of the one-dimensional problem in the direction of the normal to the median surface of the shell structure was carried out using the numerical method of discrete orthogonalisation. The present study investigates the values of critical loads of external lateral pressure for an anisotropic cylindrical shell made of a fibrous composite and the same shell with a layer of a functionally graded material. The study investigates the dependence of the critical loads on the angle of rotation of the main elastic directions of the unidirectional fibrous material and the number of its layers. The findings of this study demonstrate that incorporating a functionally graded material layer has the potential to enhance the critical loads of the structure.

Keywords: anisotropic laminated shell, fibrous composite, functionally graded material, critical load, spatial formulation

INTRODUCTION

Resolving problems in the theory of elasticity and structural mechanics is closely related to the utilisation of variational principles (Tonti, 1967; Abovsky, Andreev & Deruga, 1978; Washizu, 1982; Lanczos, 1986), which can be employed to create new consistent models for calculating shell structures. The utilisation of variational principles is also associated with the finite element method, a widely employed technique in the design of contemporary engineering structures (Zienkiewicz, 1972; Bazhenov, Kryvenko & Solovey, 2010; Kryvenko, Lizunov, Vorona & Kalashnikov, 2024). The development of classical and refined theories for calculating shell structures has been enabled by the adoption of variational methods (Grigorenko & Kryukov, 1988; Bazhenov, Semenyuk & Trach, 2010; Trach, Podvorny & Khoruzhiy, 2019). These theories have been instrumental in facilitating the acquisition of reliable solutions to problems concerning the stress–strain state and the stability and dynamics of anisotropic shells. However, the use of modern composite materials in modern engineering

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increases the demands on the construction of mathematical models and, consequently, the need for more accurate calculation approaches. For example, modern functionally graded materials (FGMs) are heterogeneous anisotropic composites consisting of different phases of material components. The widespread use of FGMs in various fields of modern engineering requires the development of approaches (Reddy, 2000; Lee, Zhao & Reddy, 2010; Kołakowski & Mania, 2012; Kowal-Michalska & Mania, 2013) that make it possible to fully consider the properties of such materials under different types of impacts. The use of FGMs as materials for shell structures requires consideration of changes in their structure, such as thickness, which can be done using the relations of the spatial theory of elasticity. A substantial corpus of works has been dedicated to resolving issues pertaining to the stability of shell structures composed of isotropic and orthotropic materials in three dimensions. Among these, the most comprehensive and generalised are the scientific works (Guz & Babich, 1980; Guz & Babich, 1985; Guz, 1986) in which the problems of the three-dimensional stability of cylindrical shells made of orthotropic materials under external forces are solved. However, the advent of modern composite materials has necessitated the refinement of methodologies for addressing these stability concerns. This discrepancy may be attributed to the disparity between the predominant elasticity orientations of the pre-orthotropic material and the curvilinear coordinate system employed in the shells (Fig. 1).

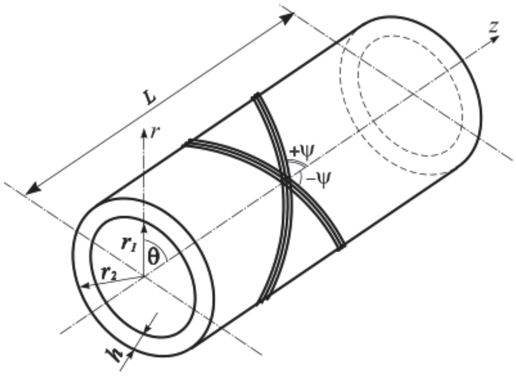

Fig. 1. Anisotropic non-thin cylindrical shell (ψ – rotation angle; z, θ, r – coordinates, r_1 – radius of the inner surface, r_2 – radius of the outer surface, L – length of the cylinder's base, h – wall thickness of the cylinder)

Source: Semenyuk, Trach and Podvornyi (2023).

The complexity of creating approaches to calculate such shell structures is caused by the interconnection of tensile (compression) and shear, bending, and torsion deformations, which leads to more complex equations of subcritical stress–strain state and stability compared to orthotropic ones (Ambartsumyan, 1974; Lekhnitskii, 1981; Trach, Semenuk & Podvornyi, 2016; Podvornyi, Semenyuk & Trach, 2017; Semenyuk, Trach & Podvornyi, 2019; Trach et al., 2019; Trach, Podvorny & Zhukova, 2023). This paper proposes an approach to modifying the functional of the generalised Hu–Washizu variational principle to a form that considers the peculiarities of modern materials and is based on the relations of the theory of elasticity of an anisotropic body. The elastic properties of these materials are confined to a single plane of elastic symmetry.

BASIC EQUATIONS. THE HU–WASHIZU PRINCIPLE OF VARIATIONS

In accordance with the Hu–Washizu variational principle (Washizu, 1982), it can be demonstrated that the equilibrium equation, stability equation, elasticity relations, geometric relations and corresponding boundary conditions can be obtained from the stationarity condition of the functional Π_1, which is defined from the integral:

$$\Pi_1 = \iiint_V \left\{ W\left(e_{ij}\right) + \Phi\left(u_i\right) - \sigma_{ij}\left[e_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right]\right\} dV + \iint_{S_1} \Psi\left(u_i\right) dS_1 - \iint_{S_2} p_i\left(u_i - \overline{u}_i\right) dS_2, \tag{1}$$

where (without additional conditions):
u_i – displacements,
e_{ij} – defined in terms of the deformation components ε_{ij} from the geometrically linear theory of elasticity,
σ_{ij} – stresses on the surface S_1,
p_i – stresses on the surface S_2 (varied), induced by the displacements $\overline{u}_i$,
$W(e_{ij})$ – potential energy of deformation,
$\Phi(u_i)$, $\Psi(u_i)$ – volumetric and surface load potentials, respectively, the semicolon before the parameters i, j denotes the covariant derivative at the coordinate with the corresponding index i, j = 1, 2, 3, where 1 corresponds to coordinate z, 2 to coordinate θ, and 3 to coordinate r (Fig. 1),
V – volume of the casing material.

The strain energy potential, as represented by the vector-matrix notation, is expressed as follows:

$$W\left(e_{ij}\right) = \frac{1}{2}\varepsilon^T B\varepsilon, \tag{2}$$

where:
$\varepsilon^T = (\varepsilon_{zz}, \varepsilon_{\theta\theta}, \varepsilon_{rr}, 2\varepsilon_{r\theta}, 2\varepsilon_{rz}, 2\varepsilon_{z\theta})$ – vector of deformation components in the geometrically linear theory of elasticity,
B – elasticity matrix.

In accordance with work by Washizu (1982), the following equations can be derived by applying the stationarity condition of $\delta\Pi_1$ and introducing the vector $\sigma^T = (\sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rr}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{z\theta})$:

1. For the elasticity relation, the equation takes the form:

$$\sigma = B\varepsilon. \tag{3}$$

2. For geometric relationships, the equation takes the form:

$$\varepsilon = \varepsilon(u). \tag{4}$$

3. For equilibrium equations, the equation takes the form:

$$\sigma_{ij,j} + f_i = 0. \tag{5}$$

Additionally, the boundary conditions $\sigma_{ij}n_j = \overline{F}_i$ on the surface S_1 as well as the displacements $u_i = \overline{u}_i$ and stresses $p_i = \sigma_{ij}n_j$ on the surface S_2 can also be obtained.

In the dependencies for deformations, (4) shows the relationship between deformations and displacements. In the inverse of the elasticity relations (3), the strain–stress dependence is represented as:

$$\varepsilon = A\sigma, \tag{6}$$

where Matrix $A = B^{-1}$.

Coefficients of the Matrix A will be denoted as a_{ij}, and the Matrix B as b_{ij} $(i, j = \overline{1,6})$. Matrices A and B are symmetrical, because $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$.

Modification of the mixed variational principle in deriving the equations of subcritical stress–strain state

To develop the modified Hu–Washizu mixed variational principle, we divide the vectors σ and ε into two parts:

$$\begin{aligned} &\sigma_1^T = (\sigma_{rr}, \tau_{r\theta}, \tau_{rz}), \qquad \sigma_2^T = (\sigma_{zz}, \sigma_{\theta\theta}, \tau_{z\theta}),\\ &\varepsilon_1^T = (\varepsilon_{rr}, \varepsilon_{r\theta}, \varepsilon_{rz}), \qquad \varepsilon_2^T = (\varepsilon_{zz}, \varepsilon_{\theta\theta}, \varepsilon_{z\theta}). \end{aligned} \tag{7}$$

The physical relationship (6) is written as a matrix:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}. \tag{8}$$

In relationship (8), the blocks A_{ij} are formed in accordance with the structure given in (7) and (8), for the case of an anisotropic material whose elastic properties are characterised by a single plane of elastic symmetry (Ambartsumyan, 1974), and taking that into account, in the case of elastic deformations, $a_{ij} = a_{ji}$:

$$\begin{aligned} &A_{11} = \begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{44} & a_{45} \\ 0 & a_{45} & a_{55} \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} a_{13} & a_{23} & a_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\\ &A_{21} = \begin{bmatrix} a_{13} & 0 & 0 \\ a_{23} & 0 & 0 \\ a_{36} & 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix}. \end{aligned} \tag{9}$$

From dependence (6), using (8), we obtain:

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}\sigma_2, \tag{10}$$

$$\varepsilon_2 = A_{21}\sigma_1 + A_{22}\sigma_2. \tag{11}$$

From equation (11), we obtain:

$$\sigma_2 = A_{22}^{-1}\varepsilon_2 - A_{22}^{-1}A_{21}\sigma_1. \tag{12}$$

Substituting (12) into (10), we obtain:

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}A_{22}^{-1}\varepsilon_2 - A_{12}A_{22}^{-1}A_{21}\sigma_1 = A_{12}A_{22}^{-1}\varepsilon_2 + \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\sigma_1. \tag{13}$$

We obtain the following from the dependence (13):

$$\sigma_1 = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \cdot \varepsilon_1 - \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \cdot A_{12}A_{22}^{-1}\varepsilon_2. \tag{14}$$

The matrix expression (3) is hereby represented in the following manner:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}. \tag{15}$$

The subsequent course of action is to be pursued in accordance with the methodology delineated in the paper by Semenyuk et al. (2023). We establish the relationship between the Matrices *A* and *B*. Then the dependence for the potential strain energy $W(e_{ij})$ can be written in the following form:

$$W_1 = W(\sigma_1, \varepsilon_2) - \sigma_{ij}\left(\varepsilon_{ij} - \varepsilon_{ij}(u)\right) =$$
$$= -\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1 - \frac{1}{2}\varepsilon_2^T\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2 + \left(\varepsilon_1^T(u) + \varepsilon_2^T(u) B_{12}^T B_{11}^{-1}\right)\sigma_1 + \varepsilon_2^T(u)\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2. \tag{16}$$

In accordance with the objective set forth in the present work, the potential of volumetric loads $\Phi(u_i)$ is neglected.

Considering the above, we write down the final form of the functionality Π_1, which is presented in (1), as follows:

$$\Pi_1 = \iiint_V \left[W(\sigma_1, \varepsilon_2)\right] dV + \iint_{S_1} \Psi(u_i)\, dS_1 - \iint_{S_2} p_i\left(u_i - \overline{u}_i\right) dS_2. \tag{17}$$

The dependence for Π_1 demonstrates the part presents the part of this functionality (1), because it replaces the number of independent variables by fulfilling the condition $\varepsilon_2 = \varepsilon_2(u)$. Taking this into account, the direct determination of the variation of the functional (17), which is caused by changes in the components of the displacement vector *u* and the stresses σ_1, takes the following form:

$$\delta\Pi_1 = \iiint_V \Bigg\{ \left[-\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1 + \left(\varepsilon_1^T(u) + \varepsilon_2(u) B_{11}^T B_{12}^{-1}\right)\sigma_1 \right]\delta\sigma_1 - \left[\frac{1}{2}\varepsilon_2^T\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2\right]\delta\varepsilon_2 +$$
$$+ \left[\varepsilon_2^T(u)\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2\right]\delta u \Bigg\} dV + \iint_{S_1}\Psi(u_i)\, dS_1 - \iint_{S_2} p_i\left(u_i - \overline{u}_i\right) dS_2. \tag{18}$$

For further transformations, we will use the geometric Cauchy relations (Novozhilov, 1961):

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z},\ \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r,\ \varepsilon_{rr} = \frac{\partial u_r}{\partial r},$$
$$\varepsilon_{z\theta} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta},\ \varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r},\ \varepsilon_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \tag{19}$$

where:

e_{rr}, e_{zz}, $e_{\theta\theta}$ – relative linear deformations in the directions of the axes of the coordinates *z*, *θ*, *r*, respectively (Fig. 1),

e_{rz}, $e_{r\theta}$, $e_{z\theta}$ – relative shear deformations tangent to the corresponding coordinate surfaces.

When considering the stationarity of the expression for the variation of the functional (18), use the dependencies for stresses $\sigma_1^T = \left(\sigma_{rr}, \tau_{r\theta}, \tau_{rz}\right)$, displacements $u^T = \left(u_r, u_\theta, u_z\right)$ and the geometric relationships (19). After equating the expressions for the independent variations of stresses $\delta\sigma_{rr}$, $\delta\tau_{r\theta}$, $\delta\tau_{rz}$ and displacements δu_r, δu_θ, δu_z in the volume integral *V* to 0, using the Euler–Lagrange method, we obtain equations describing the pre-critical stress–strain state of a cylindrical anisotropic shell in a spatial formulation:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} = -\frac{c_{23}+1}{r}\sigma_{rr} - \frac{\partial \tau_{rz}}{\partial z} - \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{c_{22}}{r^2}u_r + \frac{c_{12}}{r}\frac{\partial u_z}{\partial z} + \frac{c_{26}}{r^2}\frac{\partial u_z}{\partial \theta} + \frac{c_{26}}{r}\frac{\partial u_\theta}{\partial z} + \frac{c_{22}}{r^2}\frac{\partial u_\theta}{\partial \theta},\\
&\frac{\partial \tau_{rz}}{\partial r} = c_{13}\frac{\partial \sigma_{rr}}{\partial z} - \frac{1}{r}\tau_{rz} - \frac{c_{12}}{r}\frac{\partial u_r}{\partial z} - c_{11}\frac{\partial^2 u_z}{\partial z^2} - \frac{c_{66}}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} - \frac{c_{12}+c_{66}}{r}\frac{\partial^2 u_\theta}{\partial z \partial \theta} + \frac{c_{36}}{r}\frac{\partial \sigma_{rr}}{\partial \theta} -\\
&\qquad - \frac{c_{26}}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{2c_{16}}{r}\frac{\partial^2 u_z}{\partial z \partial \theta} - c_{16}\frac{\partial^2 u_\theta}{\partial z^2} - \frac{c_{26}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2},\\
&\frac{\partial \tau_{r\theta}}{\partial r} = \frac{c_{23}}{r}\frac{\partial \sigma_{rr}}{\partial \theta} - \frac{2}{r}\tau_{r\theta} - \frac{c_{22}}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{c_{12}+c_{66}}{r}\frac{\partial^2 u_z}{\partial z \partial \theta} - c_{66}\frac{\partial^2 u_\theta}{\partial z^2} - \frac{c_{22}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} +\\
&\qquad + c_{36}\frac{\partial \sigma_{rr}}{\partial z} - \frac{c_{26}}{r}\frac{\partial u_r}{\partial z} - c_{16}\frac{\partial^2 u_z}{\partial z^2} - \frac{c_{26}}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} - \frac{2c_{26}}{r}\frac{\partial^2 u_\theta}{\partial z \partial \theta},\\
&\frac{\partial u_r}{\partial r} = c_{33}\sigma_{rr} + \frac{c_{23}}{r}u_r + c_{13}\frac{\partial u_z}{\partial z} + \frac{c_{36}}{r}\frac{\partial u_z}{\partial \theta} + c_{36}\frac{\partial u_\theta}{\partial z} + \frac{c_{23}}{r}\frac{\partial u_\theta}{\partial \theta},\\
&\frac{\partial u_z}{\partial r} = a_{55}\tau_{rz} + a_{45}\tau_{r\theta} - \frac{\partial u_r}{\partial z},\\
&\frac{\partial u_\theta}{\partial r} = a_{45}\tau_{rz} + a_{44}\tau_{r\theta} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{1}{r}u_\theta,
\end{aligned} \tag{20}$$

where:

r – radial coordinate,

σ_{rr}, τ_{rz}, $\tau_{r\theta}$ – stress tensor components,

u_z, u_θ, u_r – movement of the shell observed in the directions corresponding to the axes of the cylindrical coordinate system z, θ, r, respectively (Fig. 1),

c_{kl} (k, l = 1, 2, 3, 6) – constant, which is the characteristics determined by the mechanical constant a_{kl} (Lekhnitskii, 1981) of the shell material:

$$\begin{aligned}
&c_{11} = \frac{1}{|A_{22}|}\left(a_{22}a_{66} - a_{26}^2\right),\ c_{12} = \frac{1}{|A_{22}|}\left(a_{16}a_{26} - a_{12}a_{66}\right),\\
&c_{22} = \frac{1}{|A_{22}|}\left(a_{11}a_{66} - a_{16}^2\right),\ c_{16} = \frac{1}{|A_{22}|}\left(a_{12}a_{26} - a_{22}a_{16}\right),\\
&c_{26} = \frac{1}{|A_{22}|}\left(a_{12}a_{16} - a_{11}a_{26}\right),\ c_{66} = \frac{1}{|A_{22}|}\left(a_{11}a_{22} - a_{12}^2\right),\\
&|A_{22}| = a_{66}\left(a_{11}a_{22} - a_{12}^2\right) + a_{26}\left(a_{12}a_{16} - a_{11}a_{26}\right) + a_{16}\left(a_{12}a_{26} - a_{22}a_{16}\right),\\
&c_{22} = \frac{1}{|A_{22}|}\left(a_{11}a_{66} - a_{16}^2\right),\ c_{16} = \frac{1}{|A_{22}|}\left(a_{12}a_{26} - a_{22}a_{16}\right),\\
&c_{26} = \frac{1}{|A_{22}|}\left(a_{12}a_{16} - a_{11}a_{26}\right),\ c_{66} = \frac{1}{|A_{22}|}\left(a_{11}a_{22} - a_{12}^2\right),
\end{aligned} \tag{21}$$

The three-dimensional system of differential equations describing the subcritical stress–strain state of anisotropic cylindrical shells in the spatial formulation and its solution are carried out according to the methodology presented in the paper by Semenyuk et al. (2023).

Modification of the mixed variational principle in the derivation of stability equations

The elastic potential (16) is represented to derive the system of differential equations of stability:

$$W_1 = -\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1 - \frac{1}{2}\varepsilon_2^T\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2 + \left(\varepsilon_1^T(u) + \varepsilon_2^T(u) B_{12}^T B_{11}^{-1}\right)\sigma_1 + $$
$$+ \varepsilon_2^T(u)\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2. \tag{22}$$

To derive the system of differential stability equations, we represent the elastic potential (16) in the form (22) and use the expansions (Novozhilov, 1961):

$$\sigma_1 = \sigma_1^0 + \alpha\sigma_1^{(1)} + \alpha^2\sigma_1^{(2)},$$
$$\varepsilon_1 = \varepsilon_1^0 + \alpha\varepsilon_1^{(1)} + \alpha^2\varepsilon_1^{(2)},$$
$$\varepsilon_2 = \varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}. \tag{23}$$

In this context, the components of the stress–strain state with a zero subscript represent the pre-critical values of deformations and stresses, obtained after solving system (20), which describes the pre-critical stress–strain state of a cylindrical anisotropic shell in a spatial formulation. On another note, those elements with an index of one are considered perturbed. Finally, components indicated by a number two remain constant but are squared; α is an infinitesimal constant that is independent of the coordinates of the adopted coordinate system of the shell structure.

After substituting (23) into (22) and performing the appropriate transformations, we obtain the following expression for the potential strain energy:

$$W_1 = -\frac{1}{2}\left(\sigma_1^0 + \alpha\sigma_1^{(1)} + \alpha^2\sigma_1^{(2)}\right)^T B_{11}^{-1}\left(\sigma_1^0 + \alpha\sigma_1^{(1)} + \alpha^2\sigma_1^{(2)}\right) - \frac{1}{2}\left(\varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}\right)^T \times$$
$$\times\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\left(\varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}\right) + \left[\left(\varepsilon_1^0 + \alpha\varepsilon_1^{(1)} + \alpha^2\varepsilon_1^{(2)}\right)^T + \left(\varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}\right)^T B_{12}^T B_{11}^{-1}\right]\times$$
$$\times\left(\sigma_1^0 + \alpha\sigma_1^{(1)} + \alpha^2\sigma_1^{(2)}\right) + \left(\varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}\right)^T \times\left(B_{22} - B_{12}^T B_{11}^{-1} B_{12}\right)\left(\varepsilon_2^0 + \alpha\varepsilon_2^{(1)} + \alpha^2\varepsilon_2^{(2)}\right). \tag{24}$$

After performing a series of mathematical transformations using equations (24) and (19) and applying the variation operation, the varied functional is obtained. By setting the coefficients of the independent variations of the displacements and stresses to zero, a linearised system of stability equations for anisotropic cylindrical shells is derived:

$$\frac{\partial\sigma_{rr}}{\partial r} = -\frac{c_{23}+1}{r}\sigma_{rr} - \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} - \frac{\partial\tau_{rz}}{\partial z} + \frac{c_{12}}{r}\frac{\partial u_z}{\partial z} + \frac{c_{22}}{r^2}u_r + \frac{c_{22}}{r^2}\frac{\partial u_\theta}{\partial\theta} + \frac{c_{26}}{r^2}\frac{\partial u_z}{\partial\theta} + \frac{c_{26}}{r}\frac{\partial u_\theta}{\partial z} +$$
$$+\left(-\frac{\partial u_z}{\partial z}c_{13} - \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right)c_{23} - \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)c_{36} - \sigma_{rr}c_{33} - \frac{2}{r}\frac{\partial u_\theta}{\partial\theta}c_{23} - \frac{1}{r}u_r c_{23} -\right.$$
$$\left.-2\frac{\partial u_\theta}{\partial z}c_{36}\right)\sigma_{rr}^0 + \left(-2\frac{\partial u_r}{\partial z}c_{23} + \frac{\partial\sigma_{rr}}{\partial z}c_{33} - \frac{\partial u_r}{\partial z}\right)\tau_{rz}^0 + \left(-\frac{2}{r}\frac{\partial u_r}{\partial\theta}c_{23} + \frac{\partial\sigma_{rr}}{\partial\theta}c_{33} - \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{1}{r}u_\theta\right)\tau_{r\theta}^0,$$

$$\frac{\partial\tau_{rz}}{\partial r} = \frac{\partial\sigma_{rr}}{\partial z}c_{13} + \frac{1}{r}\frac{\partial\sigma_{rr}}{\partial\theta}c_{36} - \frac{1}{r}\tau_{rz} - \frac{c_{12}}{r}\frac{\partial u_r}{\partial z} - \frac{c_{26}}{r^2}\frac{\partial u_r}{\partial\theta} + \left(\frac{\partial u_r}{\partial z} - \tau_{r\theta}a_{45} - \tau_{rz}a_{55}\right)\sigma_{rr}^0 +$$
$$+\left(-2r\left(\frac{\partial\tau_{r\theta}}{\partial z}a_{45} + \frac{\partial\tau_{rz}}{\partial z}a_{55}\right) - \frac{\partial u_z}{\partial z}\right)\tau_{rz}^0 + \left(-2\left(+\frac{\partial\tau_{r\theta}}{\partial\theta}a_{45} + \frac{\partial\tau_{rz}}{\partial\theta}a_{55}\right) - \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\tau_{r\theta}^0,$$

$$\frac{\partial \tau_{r\theta}}{\partial r} = \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \theta}c_{23} + \frac{\partial \sigma_{rr}}{\partial z}c_{36} - \frac{2}{r}\tau_{r\theta} - \frac{c_{22}}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{c_{26}}{r}\frac{\partial u_r}{\partial z} + \left(-\frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \tau_{r\theta}a_{44} - \right.$$
$$\left. - \tau_{rz}a_{45} + \frac{2}{r}\frac{\partial u_r}{\partial \theta}c_{23} - \frac{1}{r}u_\theta c_{23} + 2\frac{\partial u_r}{\partial z}c_{36}\right)\sigma_{rr}^0 + \left(-2r\left(\frac{1}{r}\frac{\partial u_\theta}{\partial z} + \frac{\partial \tau_{r\theta}}{\partial z}a_{44} + \frac{\partial \tau_{rz}}{\partial z}a_{45}\right) - \right.$$
$$\left. - \frac{\partial u_\theta}{\partial z}\right)\tau_{rz}^0 + \left(-2r\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}a_{44} + \frac{\partial \tau_{rz}}{\partial \theta}a_{45}\right) - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{r}u_r\right)\tau_{r\theta}^0,$$
$$\frac{\partial u_r}{\partial r} = \frac{\partial u_z}{\partial z}c_{13} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)c_{23} + \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)c_{36} + \sigma_{rr}c_{33},$$
$$\frac{\partial u_z}{\partial r} = -\frac{\partial u_r}{\partial z} + \tau_{r\theta}a_{45} + \tau_{rz}a_{55},$$
$$\frac{\partial u_\theta}{\partial r} = \frac{1}{r}u_\theta - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \tau_{r\theta}a_{44} + \tau_{rz}a_{45}. \tag{25}$$

All notations in (25) correspond to those given in (20). The stresses σ_{rr}^0, τ_{rz}^0 and $\tau_{r\theta}^0$ are determined after solving the problem of the precritical stress–strain state (20) based on the loading acting on the anisotropic cylindrical shell.

The solution of system (25) in the case of solving the stability problem under distributed external lateral pressure must satisfy the conditions on the lateral surfaces at $r = r_1$:

$$\sigma_{rr}^0(r_1, z, \theta) = 0, \quad \tau_{rz}^0(r_1, z, \theta) = 0, \quad \tau_{r\theta}^0(r_1, z, \theta) = 0$$

and $r = r_2$:

$$\sigma_{rr}^n(r_2, z, \theta) = -q_r^n(z), \quad \tau_{rz}^n(r_2, z, \theta) = 0, \quad \tau_{r\theta}^n(r_2, z, \theta) = 0. \tag{26}$$

The boundary conditions at the ends of the cylindrical shell at $z = 0$ and $z = L$ are assumed as follows:

$$\sigma_{zz} = u_r = u_\theta = 0. \tag{27}$$

Methodology for solving the stability problem

The stability problem was solved by reducing the three-dimensional stability equation system to a one-dimensional system. To reduce the three-dimensional problem (25) to a one-dimensional one, we use the procedure of the Bubnov–Galerkin analytical method (Yamaki, 1968/1969; Podvornyi et al., 2017; Semenyuk et al., 2019). According to this procedure, we decompose all the components of the stresses and displacements of system (25) into a double trigonometric series in the coordinate along the generatrix z so that they satisfy the boundary conditions at the ends of (27), and consider their periodicity along the circular coordinate θ:

$$\sigma_{rr}(r, z, \theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{1,pk}(r)\cos k\theta + y'_{1,mk}(r)\sin k\theta\right]\sin l_m z,$$
$$\tau_{rz}(r, z, \theta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{2,pk}(r)\cos k\theta + y'_{2,mk}(r)\sin k\theta\right]\cos l_m z,$$
$$\tau_{r\theta}(r, z, \theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{3,pk}(r)\sin k\theta + y'_{3,mk}(r)\cos k\theta\right]\sin l_m z,$$

$$u_r(r,z,\theta)=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{4,pk}(r)\cos k\theta+y'_{4,mk}(r)\sin k\theta\right]\sin l_m z,$$
$$u_z(r,z,\theta)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{5,pk}(r)\cos k\theta+y'_{5,mk}(r)\sin k\theta\right]\cos l_m z,$$
$$u_\theta(r,z,\theta)=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{6,pk}(r)\sin k\theta+y'_{6,mk}(r)\cos k\theta\right]\sin l_m z, \quad (28)$$

where:

$y_{i,pk}$, $y'_{i,mk}$, $(i=\overline{1,6})$ – coefficients for the expansion into trigonometric Fourier series of the stress and displacement components of the shell: σ_{rr}, τ_{rz}, $\tau_{r\theta}$, u_r, u_z, u_θ,

p, m, k – wave numbers in these rows,

$l_m=\frac{m\pi}{L}$, where L is the length of the cylinder's base (Fig. 1).

After some mathematical transformations and separating variables in equations (25) using relations (28), we obtain an infinite system of ordinary differential stability equations in the normal Cauchy form:

$$\frac{d\overline{y}}{dr}=T(r,\lambda)\overline{y},\quad T(r,\lambda)=t_{i,j}(r,\lambda),\quad i=\overline{1,\infty},\quad j=\overline{1,\infty}, \quad (29)$$

where:

$\overline{y}=\left\{y_{1,p};y_{2,p};y_{3,p};y_{4,p};y_{5,p};y_{6,p};y'_{1,mk};y'_{2,mk};y'_{3,mk};y'_{4,mk}\,y'_{5,mk};y'_{6,mk}\right\}^T$ – solution vector function composed of the coefficients of the double Fourier series expansion of the stress and displacement components in the axial direction z, and the circumferential coordinate θ of the shell is a solving vector function,

$T(r,\lambda)$ – square matrix with variable coefficients that depends on the argument r and load parameters λ.

The one-dimensional system of differential equations (29) is solved using the numerical method of discrete orthogonalisation. This method has been adapted to solve stability problems of anisotropic cylindrical shells (Bazhenov et al., 2010).

To determine the critical load values, the static Euler criterion is used, and the problem is solved by a step-by-step search method. In this process, an external load is specified, for which system (20) is solved. The components of the stress–strain state σ^0_{rr}, τ^0_{rz} and $\tau^0_{r\theta}$, determined from (20) using (15), are substituted into system (29), and the discriminant of the matrix of this system of equations is computed, taking into account the appropriate boundary conditions on the shell surfaces. This process is repeated with varying external load values until a so-called non-trivial solution is obtained, that is, when the discriminant of the matrix of system (29) becomes zero, which corresponds to the moment of loss of stability.

RESULTS AND DISCUSSION

The following presentation outlines the possibilities of the proposed approach. The problem of comparing the values of critical loads of an anisotropic cylindrical shell made of a pre-orthotropic fibrous material, boron plastic, with the same shell but having an outer layer of a ceramic-metallic functionally graded material, was considered. The FGM layer can be used as a protective layer for the fibrous material when exposed to high-intensity temperature fields. It is important to note that both shell structures are subject to distributed external pressure. The main directions of elasticity of the boron plastic and FGM can be rotated by an angle ψ relative to the direction of the shell (Fig. 1).

Mechanical characteristics of a unidirectional fibrous boron plastic composite are (Lubin, 2013): $E_{zz} = 280E_0$, $E_{\theta\theta} = E_{rr} = 31E_0$, $G_{z\theta} = G_{r\theta} = 10.5E_0$, $G_{rz} = 21.2E_0$, $\nu_{\theta z} = 0.25$, $E_0 = 1{,}000$ MPa.

A mixture of silicon nitride and titanium alloy with a volume fraction of mixed materials ($N = 1$) was selected as the FGM (Shen, 2009). The components of the FGM at the temperature of the initial undeformed state $t_0 = 293$ K have the following mechanical characteristics: silicon nitride: $E_c = 322.77E_0$, $\nu_c = 0.24$; titanium alloy (Ti-6Al-4V): $E_m = 106.09E_0$, $\nu_m = 0.298$.

The change in the volume fractions of silicon nitride and titanium alloy occurs with the thickness of the FGM layer. The general characteristics of the FGM are determined according to (Shen, 2009):

$$E(\xi) = (E_c - E_m)\left(\frac{\xi_f}{h_f}\right)^N + E_m, \quad \nu(\xi) = (\nu_c - \nu_m)\left(\frac{\xi_f}{h_f}\right)^N + \nu_m, \tag{30}$$

where:
$E(\xi)$, $\nu(\xi)$ – mechanical characteristics of the joint material in terms of thickness,
h_f – thickness of the functionally graded layer of the shell,
ξ_f – coordinate over the thickness of the FGM layer $\xi_f = r - r_{f1}$,
r – coordinate of the shell thickness of an arbitrary point in the general coordinate system (Fig. 1),
r_{f1} – coordinate of the inner surface of the FGM layer in the general coordinate system.

The cylindrical shell structure made of boron plastic has the following geometric parameters (Fig. 1): $L = 1.2$ m, $r_1 = 0.5925$ m, $r_2 = 0.6075$ m. The comparable boron plastic fibre composite and FGM shells have similar geometrical characteristics, but are composed of two types of materials in terms of thickness: the inner part $r_1 = 0.5925$ m, $r_{f1} = 0.6025$ m (made of boron plastic layer) and outer layer $r_{f1} = 0.6025$ m, $r_2 = 0.6075$ m (made of ceramic-metal FGM).

A study was conducted to assess the stability of cylindrical shell structures with a mid-surface radius $r_0 = 0.6$ m based on the ratio of the total thickness of the shells h based on their mid-surface radius $\frac{h}{r_0} = \frac{1}{40}$.

External lateral pressure was exerted on the shells $q = -q_0 \sin\left(\frac{\pi z}{L}\right)$. The temperature of the materials remains unchanged (293 K) and corresponds to the temperature of the initial undamaged state of the structures.

Table 1 and Figure 2 show the results of a comparison of the critical loads of the external side pressure (q^{cr}) from rotation angle (ψ) on the main directions of elasticity of the shell material. In Figure 2, Curve 1 (solid) represents the results obtained for the boron plastic design. In Figure 2, Curve 2 (dashed) characterises the two-layer structure of the shell: the inner layer is boron plastic; the outer layer is FGM.

Table 1. Critical loads (q^{cr}) for boron plastic shell and boron plastic shell with functionally graded materials (FGM) layer

Specification	Angle ψ [°]									
	0	10	20	30	40	50	60	70	80	90
q^{cr} for boron plastic shell [MPa]	2.36 (6)	2.67 (6)	3.34 (6)	4.04 (6)	4.79 (5)	5.65 (5)	6.58 (4)	7.07 (4)	7.61 (4)	7.90 (4)
q^{cr} for boron plastic shell with a layer of ceramic-metal FGM [MPa]	4.49 (5)	4.61 (5)	5.03 (5)	5.57 (5)	6.58 (5)	7.84 (5)	8.84 (5)	9.74 (5)	10.05 (4)	10.34 (4)
Difference between critical loads (Δ) [%]	90.3	72.7	50.6	37.9	37.4	38.8	34.3	37.8	32.1	30.9

Note: The circumferential wave number parameter, which corresponds to the moment of stability loss, is presented in parentheses.

Source: own work.

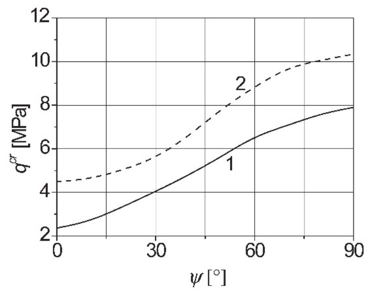

Fig. 2. Critical loads (q^{cr}) for: 1 – boron plastic shell, 2 – boron plastic shell with functionally graded materials (FGM) layer

Source: own work.

The following conclusions can be drawn based on the analysis of the results given. The use of FGM as the outer layer of a cylindrical shell structure subjected to external lateral pressure leads to an increase in critical load values over the entire range of variation of the rotation angle of the principal material directions. The greatest difference between the compared values reaches 90.3% and occurs at $\psi = 0°$. With the increasing angle ψ, the difference between the compared results decreases and is 30.9% at $\psi = 90°$. To explain this effect, in the case of distributed lateral pressure, it is possible to increase the modulus of elasticity of the fibrous material of the boron plastic in the circular direction with an increase in angle ψ in the range from 0° to 90°. It is also noted that the results presented in Figure 2 and Table 1 indicate that even in the absence of a temperature effect on the shell, the use of FGM as the outer layer of the cylindrical shell structure leads to an increase in its bearing capacity, in terms of stability.

Considering the results presented in Table 1 and Figure 2, the problem of determining the critical loads of an anisotropic cylindrical shell made of a different number of layers of fibrous boron plastic material with an outer layer of ceramic-metal functionally graded material, which is under the influence of distributed external pressure, was considered.

The study of critical loads of the anisotropic cylindrical shell was carried out considering the increase in the number of cross-reinforced layers of boron plastic and the change in angle $\pm\psi$. The range of their laying position is from 0° to 90°. The results of an orthotropic approach to the calculation of such a shell are also presented, considering that the mechanical characteristics c_{16}, c_{26}, c_{36}, a_{45} in the generalised Hooke law (Lekhnitskii, 1981; Semenyuk et al., 2023) are assumed to have zero values.

Table 2 and Figure 3 show the results of the study of the dependence of the critical values of the external side pressure q^{cr} on the rotation angle $\pm\psi$ with respect to the main directions of elasticity of the fibrous composite. In Figure 3, Curve 1 (continuous) represents the results obtained for a structure where the boron plastic layer of the shell is laid at an angle ψ. In Figure 3, Curve 2 (dashed) characterises a two-layer boron plastic structure, where the same thickness layers are cross-reinforced at an angle $\pm\psi$. In Figure 3, Curve 3 (dashed) and Curve 4 (bar-dashed) characterise, respectively, the three-layer and four-layer structure of the boron plastic component of the shell material, where the layers are also cross-reinforced at angles $\pm\psi$. In Figure 3, Curve 1′ presents the results of the orthotropic approach to calculating such a shell structure.

Table 2. Critical loads (q^{cr}) of layered boron plastic shell with functionally graded materials (FGM) layer

Specification	Angle $\pm\psi$ [°]									
	0	10	20	30	40	50	60	70	80	90
q^{cr} for single-layer boron plastic [MPa]	4.63 (5)	4.85 (5)	5.29 (5)	5.93 (5)	6.98 (5)	8.25 (5)	9.31 (5)	10.25 (5)	10.58 (4)	10.66 (4)
q^{cr} for two-layer boron plastic [MPa]	4.63 (5)	4.93 (5)	5.34 (5)	5.67 (5)	6.61 (5)	8.21 (5)	9.80 (5)	10.86 (5)	10.85 (4)	10.66 (4)
q^{cr} for three-layer boron plastic [MPa]	4.63 (5)	4.94 (5)	5.37 (5)	5.77 (5)	6.81 (5)	8.54 (5)	10.26 (5)	11.25 (4)	10.97 (4)	10.66 (4)
q^{cr} for four-layer boron plastic [MPa]	4.63 (5)	4.94 (5)	5.36 (5)	5.73 (5)	6.75 (5)	8.51 (5)	10.26 (5)	11.25 (4)	10.98 (4)	10.66 (4)
q^{cr} for eight-layer boron plastic [MPa]	4.63 (5)	4.94 (5)	5.37 (5)	5.75 (5)	6.79 (5)	8.58 (5)	10.35 (5)	11.30 (4)	11.00 (4)	10.66 (4)
Orthotropic approach	4.63 (5)	4.94 (5)	5.37 (5)	5.75 (5)	6.80 (5)	8.61 (5)	10.41 (5)	11.34 (4)	11.02 (4)	10.66 (4)

Note: The circumferential wave number parameter, which corresponds to the moment of stability loss, is presented in parentheses.

Source: own work.

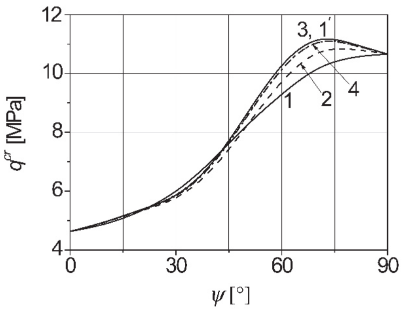

Fig. 3. Critical loads (q^{cr}) of layered boron plastic shell with functionally graded materials (FGM) layer

Source: own work.

From the analysis of the results shown in Figure 3, when the angle of rotation of the main directions of elasticity of the fibrous composite changes, the critical values of the shell with the FGM layer change. For the single-layer structure of a boron plastic component, an angle increase ψ from 0° to 90° leads to a continuous increase in the critical values of the external side pressure q^{cr}. The increase in critical loads on the change of the reinforcement angle (ψ) from 0° to 90° reaches 230%. A local maximum of critical values forms when the number of cross-reinforced layers in the fibrous component of the shell is increased: q^{cr} in the range $60° \leq \pm\psi \leq 70°$. This effect, shown in Figure 2, is most pronounced for the three-layer boron plastic structure (Curve 3), where q^{cr} exceeds the value determined at $\psi = 90°$ by 5.5%. The presented results also show that increasing the number of boron plastic layers from a single to two, in the range of rotation angle $30° \leq \pm\psi \leq 50°$, leads to a decrease in the critical values q^{cr}, with the discrepancy reaching up to 4.5%; in other ranges of $\pm\psi$, the values of q^{cr} are higher for the two-layer configuration. A further increase in the number of layers of the boron plastic package leads to the critical loads q^{cr} approaching the orthotropic approach obtained in the calculation of the structure. It should be noted that in the eight cross-reinforced layers, the discrepancy between the critical loads obtained, considering the mechanical characteristics c_{16}, c_{26}, c_{36}, and a_{45} of the generalised Hook law and without them, becomes less than 1%, so the corresponding curves in Figure 3 are not included.

CONCLUSIONS

The article presents a three-dimensional system of differential equations describing the stability of a cylindrical anisotropic shell structure, based on the Hu–Washizu variational principle. The system is reduced to a one-dimensional one by using expansions in a double trigonometric Fourier series. The approximation of the unknowns along the generative and circumferential directions of the shell structure is then carried out using the Bubnov–Galerkin analytical method. The numerical method of discrete orthogonalization is then used to solve the resulting one-dimensional system of differential equations.

The proposed approach facilitates the resolution of stability concerns inherent in the fabrication of layered shell structures utilising contemporary composite materials in a three-dimensional formulation.

A comparison of the values of critical loads of external lateral pressure for an anisotropic cylindrical shell made of fibrous composite and the same one with a layer of functionally graded material is carried out. The findings reveal that incorporating an FGM layer enhances the critical loads of the structure. The critical values of the external lateral pressure for a layered anisotropic cylindrical shell constructed from a fibrous composite with an outer layer of a functionally graded material are investigated. The investigation further involves the analysis of the critical loads in relation to the angle of rotation of the primary elastic directions of the unidirectional fibrous material, as well as the number of its layers.

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ZASADA WARIACYJNA HU–WASHIZU W ZADANIACH STATECZNOŚCI NIECIENKICH ANIZOTROPOWYCH POWŁOK CYLINDRYCZNYCH Z NOWOCZESNYCH MATERIAŁÓW KOMPOZYTOWYCH W UJĘCIU PRZESTRZENNYM

STRESZCZENIE

Z wykorzystaniem zasady wariacyjnej Hu–Washizu otrzymano trójwymiarowy układ równań różniczkowych cząstkowych teorii sprężystości ciała anizotropowego, zapisany w cylindrycznym układzie współrzędnych. Aby sprowadzić go do postaci jednowymiarowej, zastosowano analityczną metodę Bubnowa–Galerkina. Rozwiązanie jednowymiarowego zadania w kierunku normalnym do powierzchni środkowej konstrukcji powłokowej przeprowadzono przy użyciu numerycznej metody dyskretnej ortogonalizacji. Przeprowadzono badania wartości krytycznych obciążeń zewnętrznego ciśnienia bocznego dla anizotropowej powłoki cylindrycznej z kompozytu włóknistego oraz takiej samej powłoki z dodatkową warstwą materiału funkcjonalnie gradientowego (FGM). Przeanalizowano zależność wartości krytycznych obciążeń od kąta obrotu głównych kierunków sprężystości materiału jednokierunkowego oraz liczby jego warstw. Wykazano, że zastosowanie warstwy FGM pozwala na zwiększenie wartości krytycznych obciążeń konstrukcji.

Słowa kluczowe: anizotropowa powłoka warstwowa, kompozyt włóknisty, materiał funkcjonalnie gradientowy, obciążenie krytyczne, ujęcie przestrzenne