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ORIGINAL PAPER

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REMARKS ON MODELLING OF ASYMMETRIC SANDWICH BEAMS

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ABSTRACT

This paper presents two approaches to modelling the vibrations of three-layer beams with asymmetric faces. Two sets of differential equations of motion for such composite are derived. Despite sharing a common displacement hypothesis, they differ significantly. The first is based on fundamental equilibrium conditions that must be satisfied at every beam cross-section (EE). The second is the result of applying a variational approach (VAR) and minimizing the composite's energy in a global scale. In the presented computational example, both models are used to investigate the free vibration frequencies of a specific set of layered beams, and the obtained results are compared with FEM results. It should be emphasized that FEM models do not require any a priori assumed deformation hypothesis. Consequently, the presented calculations demonstrate not only the correctness and applicability of the presented modelling approaches but also the usefulness of the assumed hypothesis. The presented analysis shows that both the EE and VAR systems of motion are applicable to vibration analysis of sandwich beams in a wide range of typical design cases. As 'typical' composites one should consider asymmetric sandwich beams with relatively thin faces, where the core thickness is at least 25 times greater than thickness of any outer layers, and with any ratio of material properties of all layers. In less typical design cases, where the faces of the composite are only 10 times thinner than its core, the presented analytical solutions are still applicable to vibration analysis, but they may generate higher relative errors when compared to FEM. Based on the presented results, assessing such 'unusual' composites using VAR is not recommended. In such case the EE approach results should be considered as more predictable and closer to those obtained with FEM.

Keywords: vibrations, asymmetric sandwich beam, analytical modelling, applicability limits

INTRODUCTION

Sandwich structures are multilayered composites consisting of one or more high-strength outer layers (so-called faces) and one or more low-density inner layers (core) (Fig. 1). Each part is meant to deal with different external and/or internal factors: the core of the structure is to withstand shear loadings applied to the composite, while its faces should provide sufficient stiffness, cover the core and prevent it from punctures. Consequently, if the faces remain nearly flat, at the proper distance from each other, and they do not slide over each other, the whole composite can be characterised by a high stiffness-to-weight ratio, which is unreachable for classic homogeneous cross sections. Such structures have already found multiple applications in modern engineering, for example, in aviation (Castanie et al., 2020), naval constructions (Mouritz et al., 2001; Islami et al., 2024), spaceship constructions (Le et al., 2021), civil engineering and car design (Birman & Kardomateas, 2018; Galos et al., 2022).

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The invention of sandwich composites already has a considerable history of investigations among engineers. Let us mention several early books concerning three-layered plates by Plantema (1966), Allen (1969) or more recent by Zenkert (1995). Each of those works is a valuable contribution, which allowed further development in the theory of composites in an organised manner. Let us state that over the last decades thousands of papers concerning sandwich plates were published. In such a case, it would be impossible to follow, assess and get familiar with the state of the art if it were not for a number of significant review works by, for example, Burton and Noor (1995), Noor et al. (1996), Carrera (2000, 2001), Hu et al. (2008) or Carrera and Brischetto (2009).

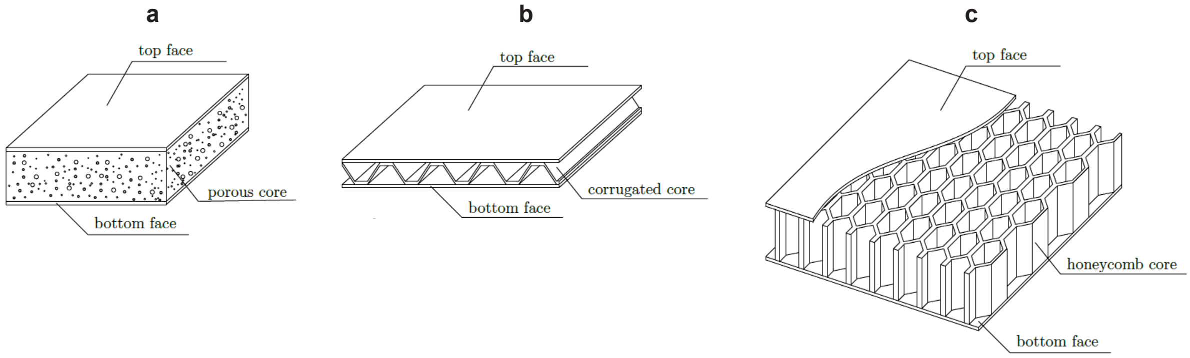

Fig. 1. Different examples of sandwich plates: a – with porous core, b – with corrugated core, c – with honeycomb core

Source: own work.

Despite years of progress, numerous issues with sandwich composites remain to be properly addressed by researchers. Let us mention work concerning the optimisation of sandwich plates with auxetic core by Gajjala and Jana (2026), dynamic response of honeycomb core sandwich plates under moving loads by Vu and Tran (2026), investigations on piercing failure by Yuan et al. (2025), examination of bandgaps in sandwich plates with corrugated core by Chen et al. (2026) or modelling of 3D-printed sandwich plates with graded pyramid lattice core by Li et al. (2026), just to mention several of the most recent works. Beyond the mentioned papers, in the literature one can find multiple other issues, including statics, vibrations, buckling, damage and fatigue analysis, delamination, impact resistance, effects of high temperature or moisture on the overall performance of the structure, and much, much more. It is easily noticeable that the variety of those issues requires many different modelling procedures, some of which are intended to be used only in very specific types of calculations. Consequently, it is crucial to invent and assess multiple methods of calculating sandwich plates. Moreover, the description of each specific procedure should be followed by an estimation of its applicability limits, so that other researchers can use it properly.

There are two original aspects of the presented paper. The first relates to the application of a special form of broken line deformation hypothesis in the investigation of the dynamics of asymmetric sandwich beams. The second focuses on the issue of the derivation of governing equations of motion. It is shown that, depending on the adopted modelling procedure, the same deformation hypothesis can lead to two different sets of equations of motion. To the author's best knowledge, such an issue has not yet been addressed in the literature. Previous works on this issue aim only at deriving the set of governing equations of motion using one selected method, but no comparison of the various available modelling methods has been made so far. Consequently, there are reports on the accuracy of multiple models of composites based on the specific deformation hypothesis, but no reports on the effect of the method of derivation of governing equations on the mentioned precision.

In the following paper, it is shown that even though those two sets of equations of motion are characterised by different coefficients, both can determine the dynamic behaviour of the same composite. Eventually, with the use of FEM formulation of the issue, the basic validation of analytical approaches is performed and several general remarks concerning the modelling of sandwich beams are formulated.

THE PROBLEM FORMULATION

The definition of the considered structure

In this section, basic assumptions of the modelling procedure are shown in detail. Let us introduce an orthogonal Cartesian coordinate system $Ox_1x_2x_3$, which describes a 3D space and t as a time coordinate. Within this space, let us consider a three-layered sandwich beam with length L_1, oriented along the x_1 axis direction. It is assumed that the midplane of the core of the structure Π lies on the plane Ox_1x_2 (Fig. 2). It can be noticed that the outer layers of the composite are asymmetric; hence, they have different thicknesses denoted as: h_f^t for the top face and h_f^b for the bottom face. The thickness of the core is equal to h_c, which eventually leads to the definition of the total thickness of the composite: $H = h_f^t + h_c + h_f^b$.

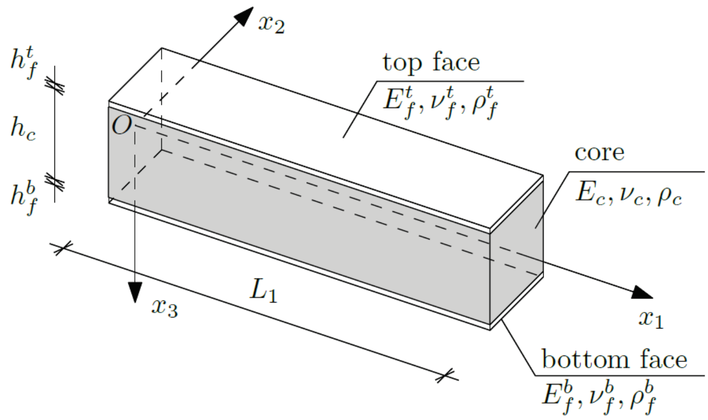

Fig. 2. A considered three-layered asymmetric beam

Source: own work.

Eventually, let us consider the material properties of each layer. In this work, it is assumed that the composite can have its layers made of different, homogeneous, isotropic materials (Fig. 2). As a consequence, the elastic moduli of each layer are denoted as: E_f^t, E_f^b and E_c, respectively, while Poisson's ratios are denoted as: v_f^t, v_f^b and v_c. Moreover, each layer can have its own, different density expressed with ρ_f^t, ρ_f^b and ρ_c. For the sake of simplicity, let us introduce one additional denotation:

$$\rho = \begin{cases} \rho_f^t & \text{for } -\dfrac{h_c}{2} - h_f^t \le x_3 < -\dfrac{h_c}{2} \\ \rho_c & \text{for } -\dfrac{h_c}{2} \le x_3 \le \dfrac{h_c}{2} \\ \rho_f^b & \text{for } \dfrac{h_c}{2} < x_3 \le \dfrac{h_c}{2} + h_f^b \end{cases}.$$

Eventually, let us state that the composite can be loaded with a certain external loading p_3 acting along the x_3 axis. It is easy to notice that such a defined structure is relatively general; hence, the described calculation case can be considered highly versatile.

In all further formulae, the spatial derivative is denoted as $\partial_i \equiv \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$, while a time derivative is denoted with an overdot.

The deformation hypothesis

The first step in modelling the dynamic behaviour of the considered sandwich beam is the application of a certain, a priori assumed deformation hypothesis. Such a hypothesis should be properly adjusted to provide reliable results while being relatively simple to use. Exemplary higher-order deformations of asymmetric sandwich beams can be found in the literature by Frostig and Shenhar (1995) or Magnucka-Blandzi et al. (2018, 2019, 2024). In this work, a variation of the broken line hypothesis, known from works of, for example, Magnucki and Szyc (2012), is used in the following form:

$$u_1(x_1, x_3, t) = \begin{cases} u_1^t(x_1, t) - \left(x_3 + \dfrac{h_c + h_f^t}{2}\right)\partial_1 w(x_1, t) & \text{for } -\dfrac{h_c}{2} - h_f^t \le x_3 < -\dfrac{h_c}{2} \\ \left(x_3 \dfrac{h_f^t + h_f^b}{2h_c} - \dfrac{h_f^t + h_f^b}{4}\right)\partial_1 w(x_1, t) + \\ -x_3 \dfrac{u_1^t(x_1, t) + u_1^b(x_1, t)}{h_c} + \dfrac{u_1^t(x_1, t) - u_1^b(x_1, t)}{2} & \text{for } -\dfrac{h_c}{2} \le x_3 \le \dfrac{h_c}{2} \\ -u_1^b(x_1, t) - \left(x_3 - \dfrac{h_c + h_f^b}{2}\right)\partial_1 w(x_1, t) & \text{for } \dfrac{h_c}{2} < x_3 \le \dfrac{h_c}{2} + h_f^b \end{cases}, \tag{1}$$

$$u_3(x_1, x_3, t) = w(x_1, t),$$

where:

$u_1(x_1, x_3, t)$ – in-plane displacement of the composite,
$u_3(x_1, x_3, t)$ – perpendicular displacement of the beam, which is constant through the whole thickness of the beam and denoted in further calculations as $w(x_1, t)$,
$u_1^t(x_1, t)$ – in-plane displacements of midplane of the top face,
$u_1^b(x_1, t)$ – in-plane displacements of midplane of the bottom face.

The physical sense of the introduced functions is presented in Figure 3.

The choice of such a hypothesis can be motivated by the fact that it uses a piecewise linear function with only three fields of displacements to model the complex behaviour of an asymmetric three-layered composite. Burton and Noor (1995) reported that similar approaches are sufficiently precise in general dynamic analysis; hence, at this stage of investigations, let us neglect all higher-order theories as those which are far more complex and difficult to use.

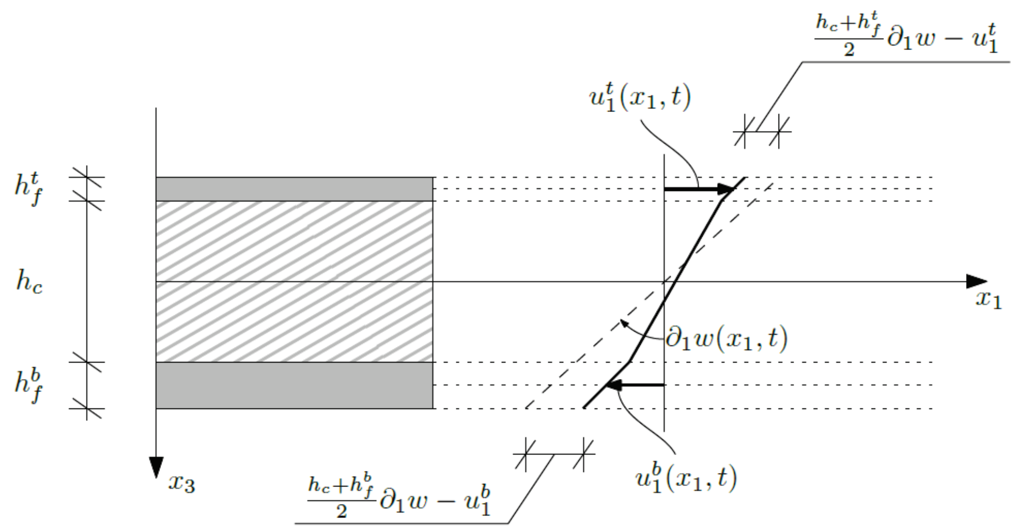

Fig. 3. A sketch of the assumed in-plane deformation hypothesis $u_1(x_1, x_3, t)$

Source: own work.

In the next step of analysis, well-known relations between displacements, strains, and stress distribution are used. Using the small deformation assumption, strains are formulated as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \quad i, j = 1,\ 3. \tag{2}$$

Consequently, the following strain definitions are obtained:

$$\begin{aligned}
&\varepsilon_{11} = \begin{cases} \partial_1 u_1^t - \left(x_3 + \dfrac{h_c + h_f^t}{2}\right)\partial_{11} w & \text{for } -\dfrac{h_c}{2} - h_f^t \le x_3 < -\dfrac{h_c}{2} \\ \left(x_3 \dfrac{h_f^t + h_f^b}{2h_c} - \dfrac{h_f^t + h_f^b}{4}\right)\partial_{11} w - x_3 \dfrac{\partial_1 u_1^t + \partial_1 u_1^b}{h_c} + \dfrac{\partial_1 u_1^t - \partial_1 u_1^b}{2} & \text{for } -\dfrac{h_c}{2} \le x_3 \le \dfrac{h_c}{2}, \\ -\partial_1 u_1^b - \left(x_3 - \dfrac{h_c + h_f^b}{2}\right)\partial_{11} w & \text{for } \dfrac{h_c}{2} < x_3 \le \dfrac{h_c}{2} + h_f^b \end{cases} \\
&\varepsilon_{33} = 0, \\
&\gamma_{13} = \begin{cases} 0 & \text{for } -\dfrac{h_c}{2} - h_f^t \le x_3 < -\dfrac{h_c}{2} \\ \left(1 + \dfrac{h_f^t + h_f^b}{2h_c}\right)\partial_1 w - \dfrac{u_1^t + u_1^b}{h_c} & \text{for } -\dfrac{h_c}{2} \le x_3 \le \dfrac{h_c}{2}, \\ 0 & \text{for } \dfrac{h_c}{2} < x_3 \le \dfrac{h_c}{2} + h_f^b \end{cases}
\end{aligned} \tag{3}$$

while stress formulation is based on Hooke's law:

$$
\sigma_{11}=\begin{cases}\dfrac{E_f^t}{1-\left(\nu_f^t\right)^2}\left[\partial_1 u_1^t-\left(x_3+\dfrac{h_c+h_f^t}{2}\right)\partial_{11}w\right] & \text{for } -\dfrac{h_c}{2}-h_f^t\le x_3<-\dfrac{h_c}{2}\\[2ex] \dfrac{E_c}{1-\left(\nu_c\right)^2}\left[\left(x_3\dfrac{h_f^t+h_f^b}{2h_c}-\dfrac{h_f^t+h_f^b}{4}\right)\partial_{11}w-x_3\dfrac{\partial_1 u_1^t+\partial_1 u_1^b}{h_c}+\dfrac{\partial_1 u_1^t-\partial_1 u_1^b}{2}\right] & \text{for } -\dfrac{h_c}{2}\le x_3\le\dfrac{h_c}{2},\\[2ex] \dfrac{E_f^b}{1-\left(\nu_f^b\right)^2}\left[-\partial_1 u_1^b-\left(x_3-\dfrac{h_c+h_f^b}{2}\right)\partial_{11}w\right] & \text{for } \dfrac{h_c}{2}<x_3\le\dfrac{h_c}{2}+h_f^b\end{cases}
$$

$$
\sigma_{33}=0, \tag{4}
$$

$$
\sigma_{13}=\begin{cases}0 & \text{for } -\dfrac{h_c}{2}-h_f^t\le x_3<-\dfrac{h_c}{2}\\[2ex] G_c\left[\left(1+\dfrac{h_f^t+h_f^b}{2h_c}\right)\partial_1 w-\dfrac{u_1^t+u_1^b}{h_c}\right] & \text{for } -\dfrac{h_c}{2}\le x_3\le\dfrac{h_c}{2}.\\[2ex] 0 & \text{for } \dfrac{h_c}{2}<x_3\le\dfrac{h_c}{2}+h_f^b\end{cases}
$$

Both stress and strain definitions are used in the subsequent subsections to derive two systems of governing equations of motion for the considered composite. It is clearly visible that the proposed deformation hypothesis is assumed in such a way that the faces of the composite behave similarly to classic Kirchhoff thin plates, while the core of the structure is the only layer capable of carrying shear stress. The physical sense behind the hypothesis is consistent with our intuitive expectations regarding the behaviour of the whole composite.

Derivation of governing equations of motion based on equations of equilibrium

In this subsection, governing equations of motion of the considered three-layered sandwich beam are derived on the basis of equations of equilibrium. Let us use the abbreviation (EE) to quickly refer to this method of derivation of equations of motion. The subsequent transformations are similar to those used to model the more general case of 2D plates, which can be found, e.g., in the work of Kączkowski (2000).

It is clear that every structure should satisfy the general equations of equilibrium, which for a 3D case can be formulated as follows:

$$
\partial_i\sigma_{ij}=\rho\ddot{u}_j,\quad \sigma_{ij}=\sigma_{ji},\quad i,j=1,\ 2,\ 3. \tag{5}
$$

Since this paper considers composite beams only, based on the system of equations (5) it is possible to obtain a system of three non-trivial equations of equilibrium:

$$
\partial_1 N_{11}-\ddot{\mu}_1=0,\quad \partial_1 M_{11}-Q_1-\ddot{\tilde{\mu}}_1=0,\quad \partial_1 Q_1-\mu\ddot{w}=-p_3, \tag{6}
$$

where:

p_3 – external loading applied to the composite and acting along the x_3 axis, while the definitions of internal forces and other denotations are as follows:

$$
\begin{aligned}
&N_{11} = \int_H \sigma_{11} dx_3, \quad Q_1 = \int_H \sigma_{13} dx_3, \quad M_{11} = \int_H \sigma_{11} x_3 dx_3, \\
&\mu = \int_H \rho dx_3, \quad \mu_1 = \int_H \rho u_1 dx_3, \quad \tilde{\mu}_1 = \int_H \rho u_1 x_3 dx_3.
\end{aligned} \tag{7}
$$

By evaluating proper integrations and proceeding with further simplifications of the set of formulae (6), one can eventually obtain the following explicit form of the equations of motion:

$$
\begin{aligned}
&\left(2D_f^t + D_c\right)\partial_{11}u_1^t - \left(2D_f^b + D_c\right)\partial_{11}u_1^b - D_c\frac{h_f^t - h_f^b}{2}\partial_{111}w + \\
&-\left[\mu_f^t\ddot{u}_1^t + \mu_c\left(-\frac{h_f^t - h_f^b}{4}\cdot\partial_1\ddot{w} + \frac{\ddot{u}_1^t - \ddot{u}_1^b}{2}\right) - \mu_f^b\ddot{u}_1^b\right] = 0, \\
&\left[D_f^t\left(h_c + h_f^t\right) + D_c\frac{h_c}{6}\right]\partial_{11}u_1^t + \left[D_f^b\left(h_c + h_f^b\right) + D_c\frac{h_c}{6}\right]\partial_{11}u_1^b - G_c\left(u_1^t + u_1^b\right) + \\
&+\left(\tilde{D}_f^t + \tilde{D}_f^b - \tilde{D}_c\frac{h_f^t + h_f^b}{2h_c}\right)\partial_{111}w + \frac{G_c}{2}\left(2h_c + h_f^t + h_f^b\right)\partial_1 w - \left(\mu_f^t\frac{h_f^t + h_c}{2}\ddot{u}_1^t + \hat{\mu}_f^t\partial_1\ddot{w}\right) + \\
&-\left[\mu_c\frac{h_c}{12}\left(\ddot{u}_1^t + \ddot{u}_1^b\right) - \hat{\mu}_c\frac{h_f^t + h_f^b}{2h_c}\partial_1\ddot{w}\right] - \left(\mu_f^b\frac{h_f^b + h_c}{2}\ddot{u}_1^b + \hat{\mu}_f^b\partial_1\ddot{w}\right) = 0, \\
&\left[D_f^t\left(h_c + h_f^t\right) + D_c\frac{h_c}{6}\right]\partial_{111}u_1^t + \left[D_f^b\left(h_c + h_f^b\right) + D_c\frac{h_c}{6}\right]\partial_{111}u_1^b + \\
&+\left(\tilde{D}_f^t + \tilde{D}_f^b - \tilde{D}_c\frac{h_f^t + h_f^b}{2h_c}\right)\partial_{1111}w - \left(\mu_f^t\frac{h_f^t + h_c}{2} + \mu_c\frac{h_c}{12}\right)\partial_1\ddot{u}_1^t - \left(\mu_f^b\frac{h_f^b + h_c}{2} + \mu_c\frac{h_c}{12}\right)\partial_1\ddot{u}_1^b + \\
&-\left(\hat{\mu}_f^t - \hat{\mu}_c\frac{h_f^t + h_f^b}{2h_c} + \hat{\mu}_f^b\right)\partial_{11}\ddot{w} + \left(\mu_f^t + \mu_c + \mu_f^b\right)\ddot{w} = p_3,
\end{aligned} \tag{8}
$$

where the following denotations are used:

$$
\begin{aligned}
&D_f^t = \frac{E_f^t h_f^t}{2\left[1-\left(\nu_f^t\right)^2\right]}, \quad D_c = \frac{E_c h_c}{2\left[1-\left(\nu_c\right)^2\right]}, \quad D_f^b = \frac{E_f^b h_f^b}{2\left[1-\left(\nu_f^b\right)^2\right]}, \\
&\tilde{D}_f^t = \frac{E_f^t\left(h_f^t\right)^3}{12\left[1-\left(\nu_f^t\right)^2\right]}, \ \tilde{D}_c = \frac{E_c\left(h_c\right)^3}{12\left[1-\left(\nu_c\right)^2\right]}, \ \tilde{D}_f^b = \frac{E_f^b\left(h_f^b\right)^3}{12\left[1-\left(\nu_f^b\right)^2\right]}, \\
&\mu_f^t = \rho_f^t h_f^t, \qquad \mu_c = \rho_c h_c, \qquad \mu_f^b = \rho_f^b h_f^b, \\
&\hat{\mu}_f^t = \rho_f^t\frac{\left(h_f^t\right)^3}{12}, \qquad \hat{\mu}_c = \rho_c\frac{\left(h_c\right)^3}{12}, \qquad \hat{\mu}_f^b = \rho_f^b\frac{\left(h_f^b\right)^3}{12}.
\end{aligned} \tag{9}
$$

The derived system of equations of motion is based on the most general idea of satisfying conditions of equilibrium by a certain stress distribution present within the composite. It should be emphasised that the initial relations between internal forces (6), together with their definitions (7), are highly versatile, as their derivation does not require any additional assumptions. Hence, it can be stated that their fulfilment should lead to a reliable model of vibrations. However, let us remind that the stress distribution functions applied to definitions (7) are the result of the a priori assumed deformation hypothesis and, as such, those functions do not have to model the behaviour of the composite properly. The performance of the presented model is shown and verified in the subsequent calculation case.

Derivation of governing equations of motion based on the variational principle

In this section, an alternative system of governing equations of motion is formulated based on the variational principle (VAR). By assuming the deformation hypothesis according to (1), after several transformations it is possible to obtain both stress (4) and strain (3) distribution functions. These functions can be used to formulate the Lagrangian ($\mathcal{L}$):

$$\mathcal{L} = U_\varepsilon - K + W, \tag{10}$$

where:

U_ε – strain energy equal to: $U_\varepsilon = \frac{1}{2}\iint\limits_{\Pi}\left(\int\limits_{H}\sigma_{11}\varepsilon_{11} + \sigma_{13}\gamma_{13}dx_3\right)dx_1 dx_2$,

K – kinetic energy equal to: $K = \frac{1}{2}\iint\limits_{\Pi}\left[\int\limits_{H}\rho\left(\dot{u}_1\dot{u}_1 + \dot{w}\dot{w}\right)dx_3\right]dx_1 dx_2$,

W – work of external forces equal to: $W = \iint\limits_{\Pi}\left(p_3 w\right)dx_1 dx_2$.

By applying the principle of stationary action to the already formulated Lagrangian (10), a set of three equations of motion is obtained. Its explicit form is as follows:

$$\begin{aligned}
&\tilde{D}_c\tilde{H}_t\partial_{111}w + \left(2D_f^t + \frac{2}{3}D_c\right)\partial_{11}u_1^t - \frac{1}{3}D_c\partial_{11}u_1^b + \frac{G_c}{h_c}\left(\hat{H}\partial_1 w - u_1^t - u_1^b\right) + \\
&-\left(\mu_f^t + \frac{1}{3}\mu_c\right)\ddot{u}_1^t - \hat{\mu}_c\tilde{H}_t\partial_1\ddot{w} + \frac{\mu_c}{6}\ddot{u}_1^b = 0, \\
&\tilde{D}_c\tilde{H}_b\partial_{111}w + \left(2D_f^b + \frac{2}{3}D_c\right)\partial_{11}u_1^b - \frac{1}{3}D_c\partial_{11}u_1^t + \frac{G_c}{h_c}\left(\hat{H}\partial_1 w - u_1^t - u_1^b\right) + \\
&-\left(\mu_f^b + \frac{1}{3}\mu_c\right)\ddot{u}_1^b - \hat{\mu}_c\tilde{H}_b\partial_1\ddot{w} + \frac{\mu_c}{6}\ddot{u}_1^t = 0, \\
&\left(\tilde{D}_f^t + \tilde{D}_c\tilde{H} + \tilde{D}_f^b\right)\partial_{1111}w + \tilde{D}_c\left(\tilde{H}_t\partial_{111}u_1^t + \tilde{H}_b\partial_{111}u_1^b\right) + \\
&-\frac{G_c}{h_c}\left[\hat{H}^2\partial_{11}w - \hat{H}\left(\partial_1 u_1^t + \partial_1 u_1^b\right)\right] + \\
&+\left(\mu_f^t + \mu_c + \mu_f^b\right)\ddot{w} - \left(\hat{\mu}_f^t + \tilde{H}\hat{\mu}_c + \hat{\mu}_f^b\right)\partial_{11}\ddot{w} - \hat{\mu}_c\left(\tilde{H}_t\partial_1\ddot{u}_1^t + \tilde{H}_b\partial_1\ddot{u}_1^b\right) = p_3,
\end{aligned} \tag{11}$$

where the denotations (9) are still held, together with the additional thickness functions:

$$\tilde{H} = \frac{\left[\left(h_f^t\right)^2 - h_f^t h_f^b + \left(h_f^b\right)^2\right]}{\left(h_c\right)^2}, \quad \hat{H} = h_c + \frac{h_f^t}{2} + \frac{h_f^b}{2}, \quad \tilde{H}_b = \frac{\left(h_f^t - 2h_f^b\right)}{\left(h_c\right)^2}, \quad \tilde{H}_t = \frac{\left(h_f^b - 2h_f^t\right)}{\left(h_c\right)^2}. \tag{12}$$

The system of governing equations of motion of the considered composite (11) is meant to satisfy the equilibrium conditions based on the energy formulation. It is easily noticeable that its form is completely different from the previously derived system of equations of motion (8), even though both are based on exactly the same deformation hypothesis (1).

CALCULATION EXAMPLE – VALIDATION OF THE APPLICABILITY LIMITS OF THE DERIVED MODELS

At this stage of problem formulation, it is vital to compare and assess the quality of results obtainable by the two previously derived models. To achieve this goal, let us perform a series of calculations of free vibration frequencies of certain three-layered asymmetric sandwich beams using both analytical solutions and the FEM model. Therefore, on the basis of similarities of presented results, approximate applicability limits of the models can be estimated.

Let us analyse a three-layered sandwich beam, simply supported on both edges. The common assumptions concerning the geometry and material properties of the composite are as follows:

$$\begin{aligned}
&h_f^t = \frac{h_c}{\zeta_1}, \qquad h_f^b = \frac{h_c}{\zeta_2},\\
&h_c = 0.1m, \qquad b = 0.1m, \qquad L_1 = 3m,\\
&E_f^t = 1{,}000 E_c, \quad E_f^b = \xi E_c,\\
&\rho_f^t = 100\rho_c, \quad \rho_f^b = 200\rho_c, \quad \upsilon_f^t = \upsilon_c = \upsilon_f^b = \upsilon,
\end{aligned}$$

where:

b – beam's width,

L_1 – beam's length,

ζ_1, ζ_2 and ξ – dimensionless coefficients, which allow variation of the properties of the composite.

Within this calculation example, the investigations of its four different geometries are performed. Their definitions are as follows:

– Structure 1: – $\zeta_1 = 100$, $\zeta_2 = 100$,

– Structure 2: – $\zeta_1 = 100$, $\zeta_2 = 50$,

– Structure 3: – $\zeta_1 = 100$, $\zeta_2 = 25$,

– Structure 4: – $\xi_1 = 100$, $\xi_2 = 10$.

It is easy to notice that Structure 1 represents the most typical, symmetric sandwich plate, the faces of which are significantly thinner than its core. All subsequent structures are asymmetric composites, whose top face is assumed to retain the same small thickness, while the thickness of the bottom face is gradually increased until it reaches 10% of the thickness of the core. The free vibrations of all four structures are examined for different values of the elastic modulus E_f^b, defined by a dimensionless parameter $\xi \in (100, 4{,}000)$.

Consequently, a variety of sandwich beams with a thicker face made of material characterised by either higher or lower material properties is considered.

For such a defined set of samples, free vibration frequencies are obtained using governing equations based on EE [cf. Eq. (8)] and VAR [cf. eq. (11)]. To derive those frequencies, a set of solutions to the issue of a simply supported beam must be implied. Please note that within the considered analytical models of layered composites there is no such thing as the angle of rotation of the cross-section understood in a similar manner as in the case of, e.g., Euler–Bernoulli beams. Consequently, the term 'simply supported' boundary conditions must be redefined. In the case of a three-layered sandwich beam, simply supported boundary conditions should be understood as the lack of perpendicular displacements on the edge of the beam and no restrictions on the in-plane displacements. The solutions, which satisfy the described conditions, can be formulated as follows:

$$w(x_1,t) = A_w \sin\left(\frac{n\pi x_1}{L_1}\right)\sin(\omega t),$$

$$u_1^t(x_1,t) = A_{u_1^t} \cos\left(\frac{n\pi x_1}{L_1}\right)\sin(\omega t),$$

$$u_1^b(x_1,t) = A_{u_1^b} \cos\left(\frac{n\pi x_1}{L_1}\right)\sin(\omega t),$$

where:

A_w, $A_{u_1^t}$, $A_{u_1^b}$ – amplitudes of displacements,
n – wave number,
ω – free vibration angular frequency.

In order to assess the results provided by both EE and VAR, a third formulation of the same issue is performed. The full 3D FEM models of each considered sandwich beam have been created using approximately 36,000 C3D8R elements, characterised by proper material properties and under the previously described 'simply supported' boundary conditions. Let us assure that in the convergence tests the mentioned number of elements proved to be sufficient to provide reliable results, while the application of reduced integration made it possible to lower the computational costs. By following the linear perturbation procedure, the free vibration frequencies of the considered set of composites are derived and treated as a benchmark solution for the EE and VAR.

In Figures 4–7, one can find relative errors between EE and VAR results of the lowest four free vibration frequencies in relation to FEM model findings, evaluated as follows:

$$K_{n,EE} = \frac{\omega_{n,FEM} - \omega_{n,EE}}{\omega_{n,FEM}}, \quad K_{n,VAR} = \frac{\omega_{n,FEM} - \omega_{n,VAR}}{\omega_{n,FEM}},$$

where:

$\omega_{n,FEM}$, $\omega_{n,EE}$, $\omega_{n,VAR}$ – n^{th} free vibration frequencies derived using a specified approach.

All presented results are evaluated for different values of elastic modulus E_f^b, defined by the dimensionless parameter ξ.

By analysing data in Figures 4–7, one can notice a respectable similarity of results. For Structures 1 and 2, the relative error between any analytically obtained result and FEM does not exceed 2%. In the case of Structure 3, the results of EE for lower values of parameter ξ tend to slightly exceed 2% relative error, but only for the third and fourth free vibration frequencies – the first two basic frequencies are still estimated with satisfying accuracy. As for VAR – all four considered frequencies of Structure 3 are approximated with an outstanding precision of less than 1.5%.

Generally, it should be noticed that for all already mentioned Structures 1–3, the VAR model seems to be characterised by slightly better similarity to FEM results than the EE modelling approach. It can be observed that in all performed calculations for all considered wave numbers n, VAR is the model that has smaller relative errors $K_{n,VAR}$.

The case of Structure 4 requires an additional comment. In Figure 7, it can be noticed that the accuracy of analytically obtained results is ambiguous. All four derived free vibration frequencies based on EE are characterised by approximately 2 ±1% relative error, which seems to decrease for structures with a more resilient thick bottom face. In the case of results of VAR, this tendency is not necessarily true, as in the extreme case of ξ = 4,000 the relative error of the fourth free vibration frequency can reach up to 8%. Let us clarify that the first two free vibration frequencies within VAR are still estimated with great accuracy.

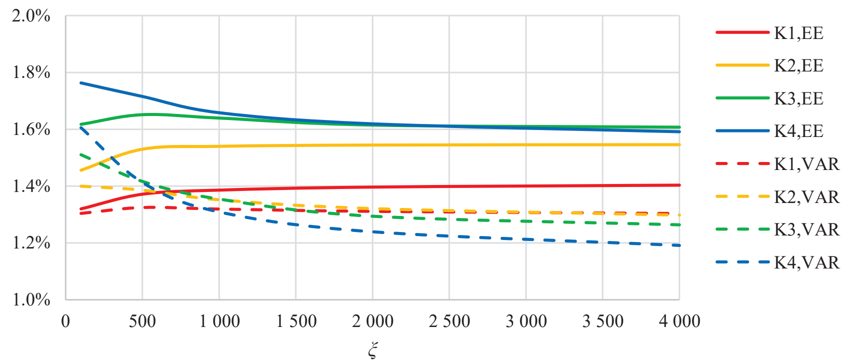

Fig. 4. Relative errors $K_{n,EE}$ and $K_{n,VAR}$ versus parameter ξ in Structure 1

Source: own work.

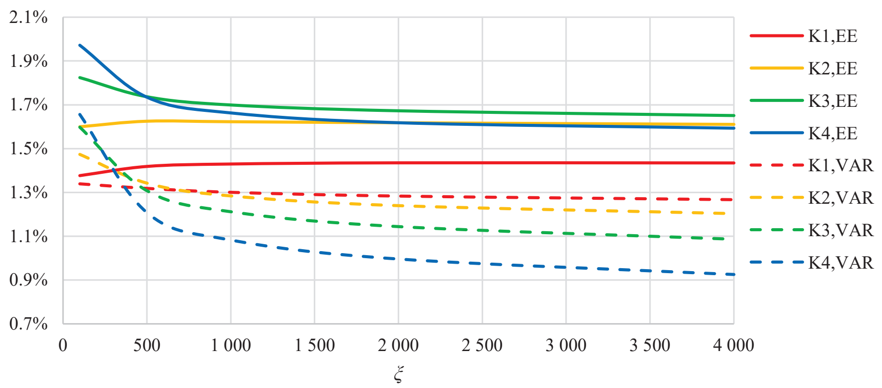

Fig. 5. Relative errors $K_{n,EE}$ and $K_{n,VAR}$ versus parameter ξ in Structure 2

Source: own work.

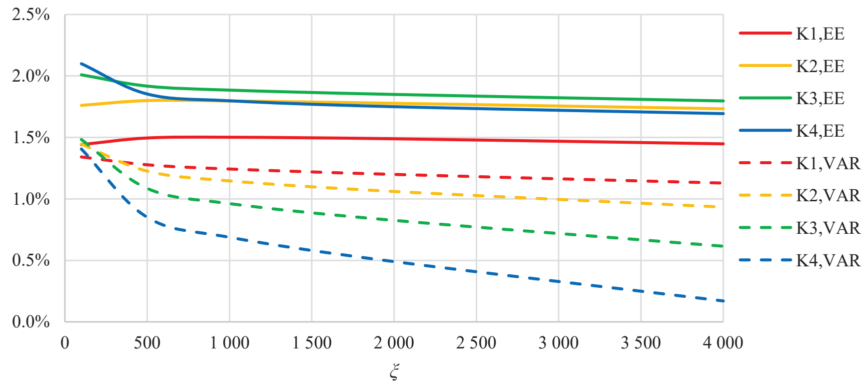

Fig. 6. Relative errors $K_{n,EE}$ and $K_{n,VAR}$ versus parameter ξ in Structure 3

Source: own work.

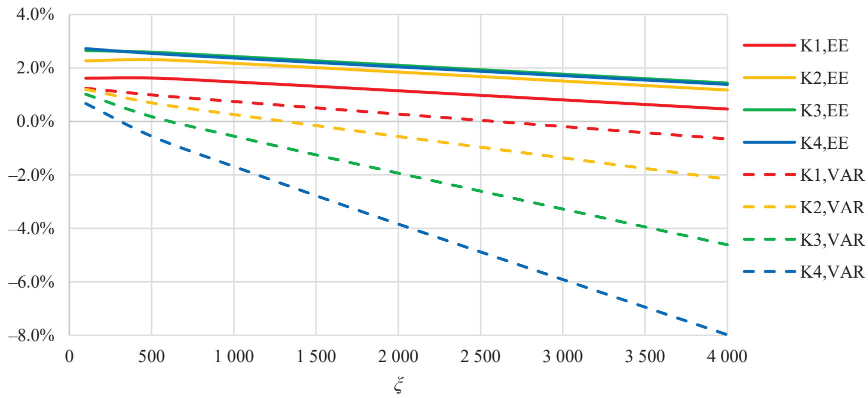

Fig. 7. Relative errors $K_{n,EE}$ and $K_{n,VAR}$ versus parameter ξ in Structure 4

Source: own work.

CONCLUSIONS

In this paper, two modelling approaches are used to derive governing equations of motion of the asymmetric three-layered sandwich beam. Despite the fact that both of these approaches are based on exactly the same deformation hypothesis, the resulting equations of motion are completely different. The first set (8) is meant to satisfy the basic equilibrium conditions in each cross-section of the beam (EE), while the second one, formulated by (11), is the result of minimisation of the energy functional (VAR).

Within the presented calculation example, both models are used to derive free vibration frequencies of a certain set of composite beams, and the obtained results are compared with a FEM formulation of the same issue. Let us emphasise that the FEM calculations do not require any a priori assumed in-plane deformation hypothesis. On the basis of the presented calculations, it is shown that the applied piecewise linear function of displacements (1) is sufficient to obtain highly precise results. Consequently, the application of higher-order deformation hypotheses (which are far more difficult to use) has lesser importance and can be simplified in the free vibration analysis. To summarise, the presented calculations prove not only the applicability of the presented modelling approaches but also the usefulness of the assumed deformation hypothesis.

As the outcome of the presented analysis, one can conclude that both EE and VAR systems of equations of motion are applicable to investigate vibrations of sandwich beams in a wide range of typical calculation cases. As ‘typical’ composites, one should perceive asymmetric sandwich beams with relatively thin faces (Structures 1–3, where the thickness of the core is at least 25 times greater than the thickness of any face) and with any proportion of mechanical properties of all layers.

In less typical calculation issues, where the faces are only 10 times thinner than the core (Structure 4), the presented analytical solutions are still applicable, but they can produce higher relative errors when compared with FEM. Consequently, due to the reported higher relative errors in the case of third and fourth free vibration frequency results, the evaluation of such ‘unusual’ composites using VAR is not recommended. However, the results of the EE approach seem to be more stable and less susceptible to higher relative errors in the case of atypical geometry of the composite.

In order to justify the above observations, let us once again emphasise the difference between EE and VAR methods. The EE method is based on the requirement of satisfying equilibrium equations (derived within the framework of the theory of elasticity in a strict and general manner) at every point of the structure. The variational principle is not so strict about this issue – its primary goal is to obtain a minimum of energy on a ‘global’ scale, which implies the possibility of certain small, local exceptions to equilibrium conditions. Consequently, different forms of governing equations are obtained within these two methods, and these two forms yield results with different precision, depending on the calculation example. As an outcome of the presented analysis, it can only be stated that in the case of composites with thicker faces, the strict fulfilment of EE is much more important than the global minimisation of energy of the composite. In other words, the exceptions to local equilibrium conditions in the VAR approach are large enough to cause noticeable discrepancies in the presented results. Further investigations of this issue, even though possible, are not within the scope of the presented paper.

Authors’ contributions

Conceptualisation: J.M.; methodology: J.M.; validation: J.M.; formal analysis: J.M.; investigation: J.M.; resources: J.M.; data curation: J.M.; writing – original draft preparation: J.M.; writing – review and editing: J.M.; visualisation: J.M.; supervision: J.M.; project administration: J.M.

All authors have read and agreed to the published version of the manuscript.

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MODELOWANIE DRGAŃ ASYMETRYCZNYCH BELEK TRÓJWARSTWOWYCH

STRESZCZENIE

W niniejszym artykule przedstawiono dwa podejścia do modelowania drgań belek trójwarstwowych o asymetrycznej budowie. Wyprowadzono tym samym dwa układy różniczkowych równań ruchu kompozytu, które pomimo łączącej je wspólnej hipotezie przemieszczeń, znacząco się od siebie różnią. Pierwszy z nich opiera się na podstawowych warunkach równowagi, które muszą być spełnione w każdym przekroju poprzecznym belki (EE). Drugi jest natomiast wynikiem zastosowania podejścia wariacyjnego i minimalizacji funkcjonału działania kompozytu (VAR). W przedstawionym przykładzie obliczeniowym oba modele wykorzystano do zbadania częstotliwości drgań własnych określonego zestawu belek warstwowych, a uzyskane wyniki porównano z wynikami MES. Należy podkreślić, że modele MES nie wymagają założenia a priori żadnej hipotezy przemieszczeń w płaszczyźnie kompozytu. W efekcie przedstawione obliczenia dowodzą nie tylko poprawności i stosowalności przedstawionych podejść modelowania, lecz także użyteczności założonej hipotezy. W wyniku przedstawionej analizy można stwierdzić, że zarówno układy równań ruchu EE, jak i VAR znajdują zastosowanie do badania drgań belek warstwowych w szerokim zakresie typowych przypadków obliczeniowych. Jako „typowe” kompozyty należy postrzegać belki warstwowe asymetryczne o stosunkowo cienkich okładzinach, gdzie grubość rdzenia jest co najmniej 25 razy większa niż grubość dowolnej powierzchni, i o dowolnym stosunku właściwości materiałowych wszystkich warstw. W mniej typowych zagadnieniach obliczeniowych, gdzie okładziny belki są tylko 10 razy cieńsze od rdzenia, przedstawione rozwiązania analityczne nadal mają zastosowanie w analizie drgań, jednak mogą generować już większe błędy względne w porównaniu z MES. Na podstawie przedstawionych wyników ocena takich „nietypowych” kompozytów z wykorzystaniem VAR nie jest zalecana. W przypadku nietypowej geometrii kompozytu wyniki podejścia EE wydają się bardziej przewidywalne i bliższe wynikom otrzymywanym w ramach MES.

Słowa kluczowe: drgania, asymetryczna belka trójwarstwowa, modelowanie matematyczne, zakres stosowalności