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REVIEW PAPER

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THE INFLUENCE OF MICROSTRUCTURE SIZE ON TEMPERATURE DISTRIBUTION IN BIPERIODIC COMPOSITE

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ABSTRACT

In this work, the problem of heat conduction in a biperiodic composite consisting of two constituent materials was analysed. To average the discontinuous coefficients in the heat conduction equation, which arise from the heterogeneous structure, the technique of tolerance modelling was applied. The finite difference method was then used to solve the resulting system of equations and to determine the distribution of the unknowns. The finite difference method algorithm was implemented in Maple 2019 software. The main objective of the analysis was to investigate the influence of the composite's microstructure size on the distribution of the unknowns, facilitated by the tolerance model equations derived through the tolerance modelling technique. The study demonstrated how the number of composite cells, and consequently the cell size, affects temperature values and their fluctuations under selected boundary conditions.

Keywords: heat conduction, composites, biperiodicity, tolerance modelling

INTRODUCTION

Composites are structures composed of two or more materials. The design of such structures aims to create materials that exhibit better properties than homogeneous materials. Composites can feature improved mechanical strength, hardness, as well as lower weight or better thermal conductivity than homogeneous structures. Composite structures are widely used across various industries due to their enhanced properties compared to traditional materials. In the aerospace industry, composites are used to manufacture lightweight yet strong components, such as aircraft fuselages, wings and engine parts, which contribute to fuel efficiency. In the automotive sector, they are employed in the production of car bodies, chassis and interior components, thereby reducing vehicle weight. In the construction industry, composites are utilised for building materials, offering durability and resistance to environmental factors. The marine industry benefits from composites in the construction of hulls and other structural parts of boats and ships, providing resistance to corrosion and reduced maintenance costs. Composites are also used in the sporting goods industry to create high-performance equipment such as tennis rackets, golf clubs and bicycles, where strength, stiffness, and lightweight are essential. Additionally, in the energy sector, composites are integral to the manufacturing of wind turbine blades and other renewable energy infrastructure, where their strength and lightness enhance efficiency and longevity.

Therefore, it is crucial to understand the methods that allow for the examination and analysis of heterogeneous structures, among which periodic and biperiodic structures can be distinguished. Among such methods, asymptotic homogenisation (Bensoussan, Lions & Papanicolay, 1978), a certain variant of

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homogenisation that introduces the concept of microlocal parameters (Matysiak & Nagórko, 1989), the finite element method (Santos, Mota Soares, Mota Soares & Reddy, 2008), the higher order theory (Aboudi, Pindera & Arnold, 1999) and tolerance modelling (Woźniak, Michalak & Jędrysiak, 2008) should be mentioned.

In the case of periodic or biperiodic structures with a repetitive (recurrent) pattern, it is possible to mentally isolate repeatable cells that have the same structure. The microstructure parameter is then defined as the dimension of such a cell for periodic structures, or as the two dimensions along two perpendicular axes of the coordinate system for biperiodic structures. Not all of the mentioned methods take into account the influence of the size of the microstructure parameter in the issues considered. Therefore, this work employs tolerance modelling, which allows for the consideration of this influence. This technique, by introducing a series of new concepts and assumptions, enables the averaging of equations where the coefficients are discontinuous and highly oscillating. Tolerance modelling is used in the analysis of heat conduction (Kubacka & Ostrowski, 2021; Ostrowski & Jędrysiak, 2021), thermoelasticity (Kubacka & Jędrysiak, 2018; Tomczyk, Gołąbczak & Gołąbczak, 2024), dynamics (Domagalski, 2018; Tomczyk, Gołąbczak, Litawska & Gołąbczak, 2023) and stability problems (Marczak & Jędrysiak, 2021; Tomczyk, Bagdasaryan, Gołąbczak & Litawska, 2021).

The problem analysed in this work is the heat conduction issue in a biperiodic composite, made of two different materials. The equations describing it were averaged using tolerance modelling, and their numerical solution was then obtained using the finite difference method. The finite difference method algorithm was implemented in the Maple 2019 software. This algorithm enabled the calculation and presentation of the temperature distribution in the analysed structure under the assumed boundary conditions. Additionally, the influence of the volume fractions of the individual materials within the cell on the temperature distribution throughout the entire composite was examined. The volume fraction of the individual materials affects not only the thermal properties of the structure but also its weight, mechanical strength and production cost.

OBJECT UNDER CONSIDERATION

Figure 1 shows the analysed composite, consisting of two different constituent components. The dimensions of the composite are denoted by Lx along the x-axis and Ly along the y-axis. The microstructure parameter (mentally isolated cell dimension) along the x-axis is denoted by lx and along the y-axis is denoted by ly and it is dependent on the number of cells (N) in the entire structure.

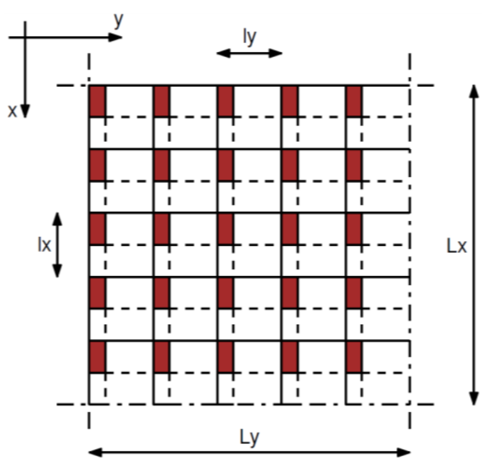

Fig. 1. Composite with biperiodic structure

Source: own work.

The volume fraction of the first and second material within the cell is constant. In the conducted analysis, the number of cells (N) is variable and, consequently, the dimension of the microstructure parameters are also variable.

In relation to the considered structure, the heat conduction problem was analysed and described by following Fourier equation:

$$\partial_i(\boldsymbol{K}_{ij}\partial_j\theta) - c\rho\dot{\theta} = 0, \tag{1}$$

where:
$\boldsymbol{K}_{ij}$ – tensor of conductivity,
θ – total temperature field,
c – specific heat,
ρ – mass density,
i, j – indices equal to 1, 2, 3.

The material coefficients in Eq. 1 are discontinuous due to the heterogeneous structure of the composite.

TOLERANCE MODELLING

Tolerance modelling is a technique initiated by Professor Czesław Woźniak and later continued and developed in many research institutes (Tomczyk, Gołąbczak, Kubacka, Bagdasaryan, 2024). In addition to allowing for the consideration of the influence of microstructure size in the analysed problems, this technique does not require solving the problem at the cell level to determine appropriate shape functions – unlike asymptotic homogenisation.

This technique introduces a series of new concepts, the most significant of which, in the context of the conducted research, are the tolerance-periodic function, the slowly varying function and the highly oscillating function.

A function f can be described as tolerance-periodic (with respect to the primary cell Δ and the tolerance parameter δ) if it satisfies the following conditions:

$$(\forall x \in \Omega)\left(\exists \tilde{f}^{(i)}(x,\cdot) \in H^0(\Delta)\right)\left(\left\|\partial^i f|_{\Omega x}(\cdot) - \tilde{f}^{(i)}(x,\cdot)\right\|_{H^0(\Omega x)} \le \delta\right), \tag{2}$$

$$\int_{\Delta(\cdot)} \tilde{f}^{(i)}(\cdot,z)dz \in C^0(\bar{\Omega}), \tag{3}$$

where:
$\partial^i f$ – gradient of order i function f,
$f(x,\cdot)$ with ~ – periodic approximation of the gradient $\partial^i f$,
Ω – region bounded in R^2,
Ω_x – cluster of adjacent cells,
Δ – primary cell,
$H^0(\Delta)$ – area of Δ-periodic square-integrable functions defined in R^2,
$H^0(\Omega_x)$ – area of Δ-periodic square-integrable functions defined in Ω_x,
δ – tolerance parameter.

On the other hand, function u can be described as slowly varying (with respect to the primary cell Δ and the tolerance parameter δ) when it is a tolerance-periodic and it satisfies the following condition:

$$(\forall x \in \Omega)\left(\tilde{u}^{(i)}(x,\cdot)\big|_{\Delta(x)} = \partial^i u(x),\ i = 0,1\right), \tag{4}$$

where:
$\Delta(x)$ – cell centred at the point x.

The function h can be described as highly oscillating (with respect to the primary cell Δ and the tolerance parameter δ) when it is a tolerance-periodic and it satisfies the following condition:

$$(\forall x \in \Omega)\left(\tilde{h}^{(i)}(x,\cdot)\big|_{\Delta(x)} = \partial^i \tilde{h}(x),\ i = 0,1\right). \tag{5}$$

The main assumption of tolerance modelling is the assumption of so-called micro-macro decomposition of the temperature field (in the carried-out analysis). This assumption is described by the following equation:

$$\theta(x, y) = \vartheta(x, y) + g_a(x, y) \cdot \psi_a(x, y), \tag{6}$$

where:
θ – total temperature field,
ϑ – average temperature,
g_a – fluctuation shape functions,
ψ_a – fluctuation amplitudes,
a – index runs over 1 and 2.

The average temperature ϑ and fluctuation amplitudes ψ_a are new fundamental unknowns, while the fluctuation shape functions are taken based on literature and experience. The fluctuation shape functions g_1 and g_2 adopted in this problem are a combination of saw-type function and piecewise parabolic function (Fig. 2).

Symbols v_x and v_y represent the volume fraction of the first material in the cell along the x-axis and y-axis, respectively.

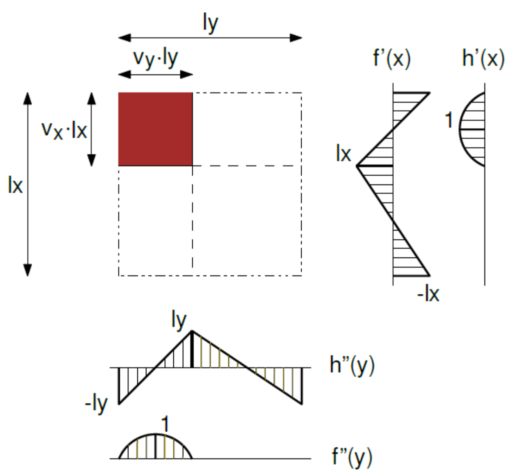

Fig. 2. Fluctuation shape functions

Source: own work.

The mentioned shape functions are expressed by the following equations:

$$g_1(x,y) = f^{'}(x) \cdot f^{''}(y), \tag{7}$$

$$g_2(x,y) = h^{'}(x) \cdot h^{''}(y). \tag{8}$$

Using the above concepts and assumptions, the equations of the tolerance model were derived, in which the coefficients are average and slowly varying. These equations include terms that directly depend on the dimension of the microstructure, as shown:

$$\langle c\rho\rangle\dot{\vartheta} + \langle c\rho g_1\rangle\dot{\psi}_1 + \langle c\rho g_2\rangle\dot{\psi}_2 - \nabla\cdot\begin{pmatrix}\langle\mathbf{K}\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g_1\rangle\psi_1 + \langle\mathbf{K}g_1\rangle\cdot\overline{\nabla}\psi_1 + \\ +\ \langle\mathbf{K}\cdot\partial g_2\rangle\psi_2 + \langle\mathbf{K}g_2\rangle\cdot\overline{\nabla}\psi_2\end{pmatrix} = 0, \tag{9}$$

$$\begin{gathered}\langle c\rho g_1\rangle\dot{\vartheta} + \langle c\rho g_1 g_1\rangle\dot{\psi}_1 + \langle c\rho g_1 g_2\rangle\dot{\psi}_2 + \langle\mathbf{K}\cdot\partial g_1\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g_1\partial g_1\rangle\psi_1 + \langle\mathbf{K}\cdot\partial g_1 g_1\rangle\cdot\overline{\nabla}\psi_1 + \\ +\ \langle\mathbf{K}\cdot\partial g_1\partial g_2\rangle\psi_2 + \langle\mathbf{K}\cdot\partial g_1 g_2\rangle\cdot\overline{\nabla}\psi_2 - \overline{\nabla}\cdot\begin{pmatrix}\langle\mathbf{K}g_1\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g_1 g_1\rangle\psi_1 + \langle\mathbf{K}g_1 g_1\rangle\cdot\overline{\nabla}\psi_1 + \\ +\ \langle\mathbf{K}\cdot\partial g_2 g_1\rangle\psi_2 + \langle\mathbf{K}g_1 g_2\rangle\cdot\overline{\nabla}\psi_2\end{pmatrix} = 0,\end{gathered} \tag{10}$$

$$\begin{gathered}\langle c\rho g_2\rangle\dot{\vartheta} + \langle c\rho g_2 g_1\rangle\dot{\psi}_1 + \langle c\rho g_2 g_2\rangle\dot{\psi}_2 + \langle\mathbf{K}\cdot\partial g_2\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g_1\partial g_2\rangle\psi_1 + \langle\mathbf{K}\cdot\partial g_2 g_1\rangle\cdot\overline{\nabla}\psi_1 + \\ +\ \langle\mathbf{K}\cdot\partial g_2\partial g_2\rangle\psi_2 + \langle\mathbf{K}\cdot\partial g_2 g_2\rangle\cdot\overline{\nabla}\psi_2 - \overline{\nabla}\cdot\begin{pmatrix}\langle\mathbf{K}g_2\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g_1 g_2\rangle\psi_1 + \langle\mathbf{K}g_1 g_2\rangle\cdot\overline{\nabla}\psi_1 + \\ +\ \langle\mathbf{K}\cdot\partial g_2 g_2\rangle\psi_2 + \langle\mathbf{K}g_2 g_2\rangle\cdot\overline{\nabla}\psi_2\end{pmatrix} = 0,\end{gathered} \tag{11}$$

where:
∇ – gradient in x-, y- and z-directions,
overlined ∇ – gradient in z-direction,
∂ – gradient in x- and y-directions.

FINITE DIFFERENCE METHOD ALGORITHM

To find the numerical solution of the tolerance model equations (Eqs 9–11), the finite difference method was chosen. The idea of this method is to replace the set of admissible solutions with a set of function values at points within the region, subject to appropriate discretisation and to replace the function derivatives at those points with the corresponding finite difference quotients defined at those points. For this purpose, the composite area was discretised in both directions (Fig. 3). Along the x-axis, it was divided into segments Δx of equal length, introducing $m + 1$ nodes, with Δx equal to $Lx \cdot m^{-1}$. Along the y-axis, it was divided into segments Δy of equal length, introducing $n + 1$ nodes, with Δy equal to $Ly \cdot n^{-1}$.

The concept of the finite difference method allows for the replacement of a system of differential equations with a system of linear algebraic equations, written at each node of the discretised region, where the unknown values of the average temperature and fluctuation amplitudes – those not specified by boundary conditions – are sought.

As mentioned, the equations are written only at the nodes where the unknowns are not defined. Unfortunately, in the analysed problem, there are three unknowns for which different boundary conditions are specified (at different nodes). Additionally, the equations written at the boundary and near-boundary nodes lead to the emergence of unknowns defined at nodes outside the composite region – so-called virtual nodes. Subsequently, the unknowns at the virtual nodes are eliminated from the equations using the boundary conditions.

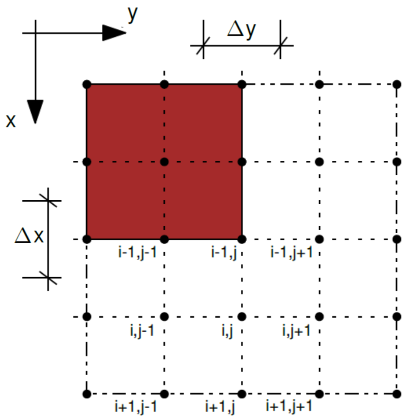

Fig. 3. Discretised area

Source: own work.

To create an algorithm in Maple 2019 software, the coefficients associated with the unknowns in all equations were grouped and ordered, forming a global coefficient matrix. Similarly, the components of the free term vector, resulting from the boundary conditions and known values of the unknowns at selected nodes, were grouped. The system of algebraic equations was then solved, yielding the numerical solution and the distribution of the sought unknowns.

EXAMPLE OF APPLICATION

The finite difference method algorithm was used to investigate the influence of microstructure size on the heat conduction analysis in a biperiodic composite with dimensions $Lx = Ly = 1$ m. It was assumed that the composite consists of two components, for which material properties such as specific heat, mass density and thermal conductivity coefficients were defined, similar to those of steel and aluminium (Fig. 4).

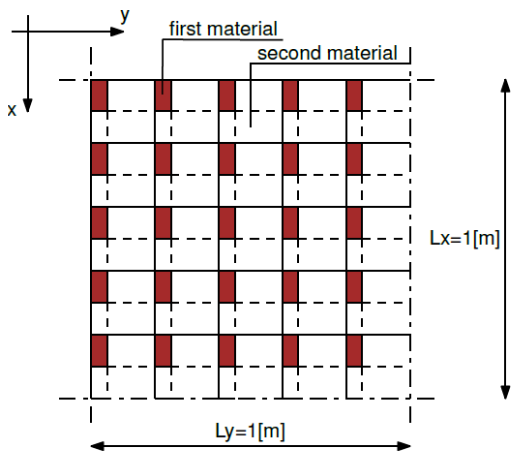

Fig. 4. Considered biperiodic composite

Source: own work.

The volume fraction of the first material in the cell was set to 0.5 along the x-axis and 0.25 along the y-axis. In the initial analysis, it was assumed that the composite consisted of $N = 10$ repeating cells along both axes of the coordinate system. Subsequently, the necessary initial boundary conditions were applied.

For the problem under consideration, the following boundary conditions were assumed: with respect to the average temperature ϑ (Fig. 5a), it was assumed that its values were known on the left and top edges of the composite (Area 1), while the right and bottom edges were thermally insulated (Area 2). With respect to the fluctuation amplitudes of the temperatures ψ_1 and ψ_2 (Fig. 5b), it was assumed that its values were known on all edges (Area 1). An initial condition was also specified in the form of known averaged temperature and fluctuation amplitudes of the temperatures during the first-time step.

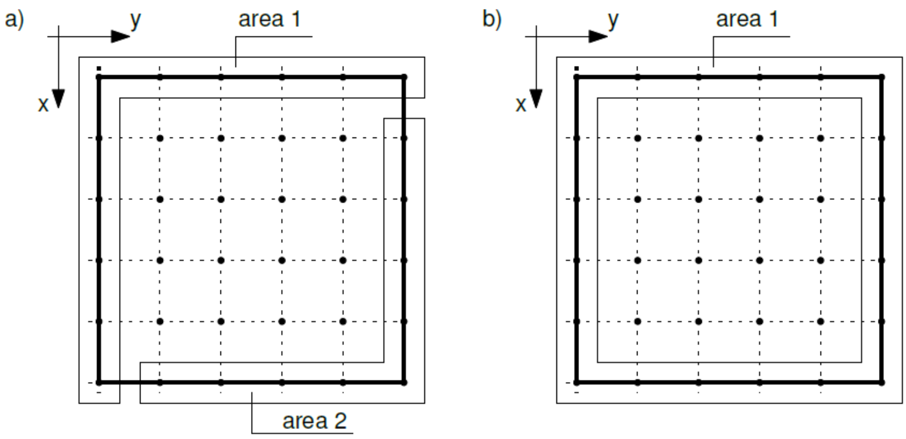

Fig. 5. Areas of defined boundary conditions

Source: own work.

The results of the initial analysis, in the form of 3D maps of the sought dimensionless unknowns (approximated) in the final time step, are shown in Figures 6–8.

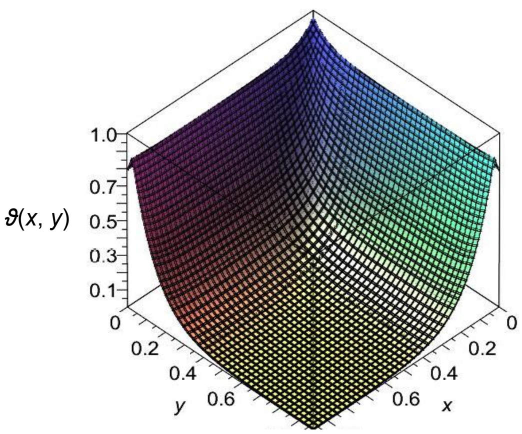

Fig. 6. Average temperature ϑ – initial analysis

Source: own work.

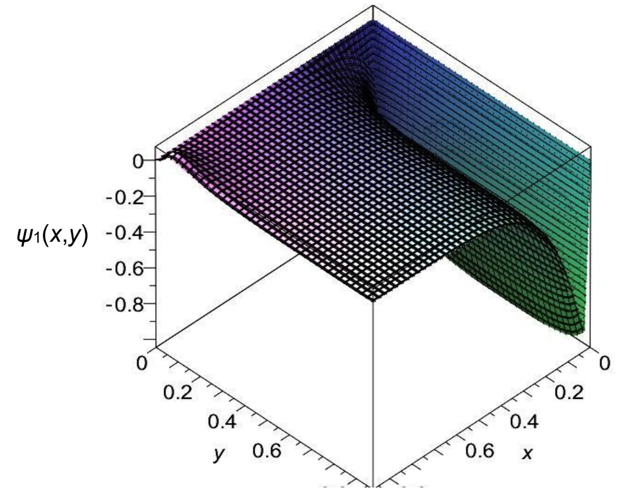

Fig. 7. Fluctuation amplitudes of the temperature ψ_1 – initial analysis

Source: own work.

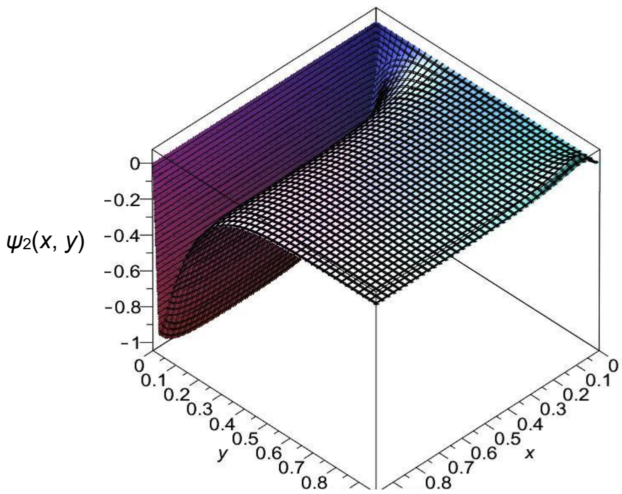

Fig. 8. Fluctuation amplitudes of the temperature ψ_2 – initial analysis

Source: own work.

The calculations were then repeated, increasing the number of cells to $N = 20$ and $N = 30$, thereby changing the dimension of the microstructure parameter.

The comparison of the average temperature distribution and the temperature fluctuation amplitudes in a selected cross-section ($x = 0.5Lx$) for different numbers of cells is presented in Figures 9–11.

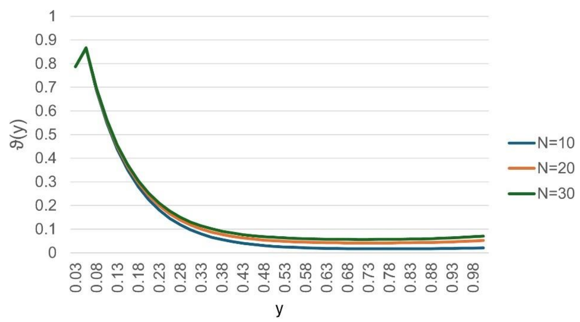

Fig. 9. Comparison of macrotemperature ϑ distribution for different numbers of cells

Source: own work.

Increasing the number of cells, and consequently reducing the microstructure size, led to an increase in the average temperature values, with relative differences (compared to the higher value) reaching up to 70.6%.

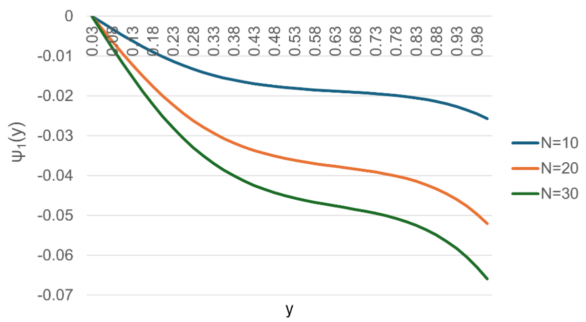

Fig. 10. Comparison of fluctuation amplitudes ψ_1 distribution for different numbers of cells

Source: own work.

A similar relation was observed in the case of fluctuation amplitudes of the temperature ψ_1, with relative differences (compared to the higher value) ranging from 49.2% to 61.1%.

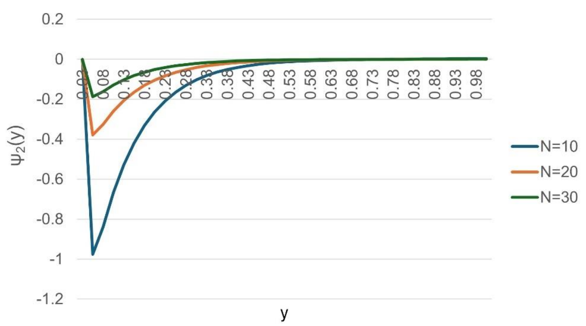

Fig. 11. Comparison of fluctuation amplitudes ψ_2 distribution for different numbers of cells

Source: own work.

An opposite relation was observed in the case of fluctuation amplitudes of the temperature ψ_2 – as the number of cells increased, meaning the microstructure parameter size reduced, the amplitude values decreased with relative differences (compared to the higher value) ranging from 40.1% to 139.5%.

SUMMARY AND CONCLUSIONS

Based on the conducted analysis, the following general and specific conclusions were formulated:

1. Tolerance modelling enables the replacement of differential equations with tolerance-periodic, discontinuous and highly oscillating coefficients, with a system of equations that have slowly varying coefficients.
2. The obtained tolerance model equations allow for the consideration of the influence of microstructure size in the analysed problems.
3. The tolerance modelling technique does not require solving the problem at the cell level to find the fluctuation shape functions.
4. The created finite difference method algorithm allows for obtaining the temperature distribution in the analysed structure, which enables a better understanding of the behaviour of composites under thermal load.
5. The analysis shows the significant impact of the number of composite cells on the average temperature and fluctuation amplitudes distribution, as it directly affects the cell dimension, which corresponds to the microstructure size.

Authors' contributions

Conceptualisation: E.K. and P.O.; methodology: E.K. and P.O.; validation: E.K.; formal analysis: E.K.; investigation: P.O.; resources: E.K.; data curation: P.O.; writing – original draft preparation: E.K.; writing – review and editing: E.K. and P.O.; visualisation: E.K.; supervision: P.O.; project administration: P.O.

All authors have read and agreed to the published version of the manuscript.

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WPŁYW WYMIARU MIKROSTRUKTURY NA ROZKŁAD TEMPERATURY W BIPERIODYCZNYM KOMPOZYCIE

STRESZCZENIE

W niniejszej pracy przeanalizowano zagadnienie przewodzenia ciepła w biperiodycznym kompozycie składającym się z dwóch materiałów składowych. W celu uśrednienia nieciągłych współczynników w równaniu przewodzenia ciepła, wynikających z niejednorodnej budowy struktury, zastosowano technikę tolerancyjnego modelowania. Następnie wykorzystano metodę różnic skończonych do rozwiązania otrzymanego układu równań i znalezienia rozkładu poszukiwanych niewiadomych. Algorytm metody różnic skończonych zaimplementowano w programie Maple 2019. Głównym celem przeprowadzonej analizy było zbadanie wpływu wymiaru mikrostruktury kompozytu na rozkład poszukiwanych niewiadomych, co umożliwiły równania modelu tolerancyjnego otrzymane dzięki wykorzystaniu techniki tolerancyjnego modelowania. W pracy pokazano, w jaki sposób liczba komórek kompozytu (a co za tym idzie wymiar komórki) wpływa na wartości temperatury i jej fluktuacji w wybranych warunkach brzegowych.

Słowa kluczowe: przewodzenie ciepła, kompozyty, biperiodyka, tolerancyjne modelowanie